Faithfully flat descent

Wouter Zomervrucht, November 6, 2013

1. Faithfully flat morphisms

A ring homomorphism $A \to B$ is *faithfully flat* if for every sequence $N' \to N \to N''$ of A-modules we have that $N' \to N \to N''$ is exact if and only if $N' \otimes_A B \to N \otimes_A B \to N'' \otimes_A B$ is exact.

Proposition (Stacks 00HQ). *Let $A \to B$ be a flat ring homomorphism. Then $A \to B$ is faithfully flat if and only if* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is surjective.*

A morphism of schemes is called *faithfully flat* if it is flat and surjective.

2. Descent

'Descent' is (informally and very generally) the act of producing objects on some scheme S by doing so on several schemes U_i over S in a compatable way, and then proving that there exists a corresponding object on S. The intuition should come from Zariski descent, where the U_i are simply open subschemes of S, and descent is called *gluing*. Examples:

- A morphism $f\colon S \to X$ can be given by morphisms $f_i\colon U_i \to X$, where $\bigcup_{i\in I} U_i$ is an open covering of S, if the f_i's agree on intersections;
- A quasi-coherent sheaf $\mathcal{F}$ on S can be given by quasi-coherent sheaves $\mathcal{F}_i$ on U_i that agree on the intersections;
- A scheme X over S can be given by compatable schemes X_i over U_i.

What if we generalize *open coverings* to 'coverings' of another kind? The questions still make sense (replace intersections by fiber products).

Definition. Let S be a scheme and $\{h_i\colon U_i \to S\}_{i\in I}$ a family of morphisms to S.

- We call $\{h_i\colon U_i \to S\}_{i\in I}$ a *Zariski covering* of S if each h_i is an open immersion and $\bigcup_{i\in I} \operatorname{im} h_i = S$.
- We call $\{h_i\colon U_i \to S\}_{i\in I}$ an *étale covering* of S is each h_i is étale and $\bigcup_{i\in I} \operatorname{im} h_i = S$.
- We call $\{h_i\colon U_i \to S\}_{i\in I}$ an *fppf covering* ('fidèlement plat et de présentation finie') if each h_i is flat and locally of finite presentation and $\bigcup_{i\in I} \operatorname{im} h_i = S$.
- We call $\{h_i\colon U_i \to S\}_{i\in I}$ an *fpqc covering* ('fidèlement plat et quasi-compact') if each h_i is flat and every quasi-compact open in S is the image of a quasi-compact open under $h\colon \coprod_{i\in I} U_i \to S$.

Lemma (Stacks 0216, 021N, 022C). *These classes of coverings are increasingly general.*

Observe that $\{U_i \to S\}_{i\in I}$ is an étale/fppf/fpqc covering if and only if $\{\coprod_{i\in I} U_i \to S\}$ is an étale/fppf/fpqc covering. More examples of fpqc coverings include: $U \to S$ faithfully flat and quasi-compact; $U \to S$ faithfully flat and universally open.

3. Descent of quasi-coherent sheaves

For a covering $\{U_i \to S\}_{i\in I}$ write $U_{ij} = U_i \times_S U_j$ and $U_{ijk} = U_i \times_S U_j \times_S U_k$.

Definition. Let S a scheme and $\mathcal{U} = \{U_i \to S\}_{i \in I}$ an fpqc covering. A *descent datum of quasi-coherent sheaves* relative to $\mathcal{U}$ consists of a quasi-coherent sheaf $\mathcal{F}_i \in \operatorname{QCoh} U_i$ for all $i \in I$, and an isomorphism $\varphi_{ij} \colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$ for all $i, j \in I$, such that $\varphi_{ik}|_{U_{ijk}} = \varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}}$ (the *cocycle condition*) holds for all $i, j, k \in I$.

Definition. A descent datum $(\mathcal{F}_i, \varphi_{ij})$ *descends*, or is *effective*, if there exists a quasi-coherent sheaf $\mathcal{F} \in \operatorname{QCoh} S$ and isomorphisms $\alpha_i \colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such that $\varphi_{ij} \circ \alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$.

The following theorem is due to Grothendieck, and may be stated as: 'QCoh is an fpqc stack'.

Theorem (Stacks 023T). *Every fpqc descent datum of quasi-coherent sheaves is effective.*

The main ingredients in the proof are:
- (Stacks 022H) It suffices to prove the theorem for Zariski coverings and coverings given by a faithfully flat map $\operatorname{Spec} B \to \operatorname{Spec} A$.
- (Stacks 02JY) If $A \to B$ is faithfully flat, then any subset of $\operatorname{Spec} A$ is open if and only if its full preimage in $\operatorname{Spec} B$ is open.
- (Stacks 023M) If $A \to B$ is faithfully flat, then it is the equalizer of the maps $B \to B \otimes_A B$ given by $b \mapsto 1 \otimes b$ and $b \mapsto b \otimes 1$, respectively.

The same is true for coherent sheaves, but not for arbitrary modules.

4. Descent of morphisms

Again due to Grothendieck: 'representable functors are fpqc sheaves'.

Theorem (Stacks 023Q). *Let $\{U \to S\}$ be an fpqc covering and X a scheme over S. Suppose a morphism $f \colon U \to X \times_S U$ is such that its two pullbacks to $U \times_S U \to X \times_S U \times_S U$ coincide. Then there exists a unique morphism $g \colon S \to X$ with pullback f.*

Many properties of f 'descend' to g: injective, surjective, open, closed, separated, (locally) of finite type, (locally) of finite presentation, flat, smooth, étale, ... Not: (quasi-)projective.

5. Descent of schemes

Just a warning (Stacks 08KF): étale/fppf/fpqc descent of schemes is *not* always effective!