Géométrie Algébrique et Géométrie Analytique

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In these notes, we give a quick introduction to the main classical results in the theory of "Géométrie Algébrique et Géométrie Analytique" (GAGA). GAGA was first introduced by Jean-Pierre Serre in the 1950s, as a comparison between the category of projective integral $\mathbb{C}$-schemes of finite type (projective algebraic varieties) and the category of projective $\mathbb{C}$-manifolds (possibly with singularities). In [SGA I, Exposé XII], Grothendieck extended the theory to proper $\mathbb{C}$-schemes locally of finite types, and an extension of the concept of manifolds, called analytic spaces. He proved a series of comparison results (for example, between all the common properties of the two objects) which show that they can be, in almost all cases, regarded as the same objects. He also showed that the category of finite étale coverings of a $\mathbb{C}$-scheme locally of finite type is equivalent to that of finite étale coverings of the corresponding analytic space (Riemann's existence theorem), thus establishing a covering space theory for $\mathbb{C}$-schemes and justifying the definition of the étale fundamental group. In later talks, we will extend this justification to a more general class of schemes.

These notes give a brief summary of these topics and are not meant to be complete in any way. Interested readers can find more detailed proofs and references in the original papers by Serre [GAGA] and Grothendieck [SGA I, Exposé XII].

1 Analytic spaces

1.1 Definitions and some basic facts

Consider the space $\mathbb{C}^n$. Let $U \subset \mathbb{C}^n$ be an open subspace and $\mathcal{H}$ be the sheaf of holomorphic functions on U.

Definition 1.1. Let $f_1, \ldots, f_k$ be holomorphic functions on U and $\mathcal{I}$ be the coherent sheaf of ideals generated by these functions. An **affine analytic space** $(X, \mathcal{H}_X)$ is a locally ringed space whose underlying space is

$$X = \{y \in U | f_1(y) = \cdots = f_k(y) = 0\} \subset U$$

and whose sheaf of rings is defined by $i^{-1}\mathcal{H}/\mathcal{I}$ where $i : X \to U$ is the inclusion map. The stalk at each point $x \in X$ is given by $\mathcal{H}_{X,x} = \mathcal{H}_x/(f_1, \ldots, f_k)$.

An **analytic space** $(X, \mathcal{H}_X)$ is a locally ringed space satisfying

(i) there is an open cover $\{V_i\}$ of X such that each $(V_i, \mathcal{H}_X|_{V_i})$ is an affine analytic space; and

(ii) X is a separated (Hausdorff) topological space.

A morphism of analytic spaces is a morphism of locally ringed spaces.

Exercise 1.2. Check that the gluing of the open affine subspaces is well-defined.

Remark 1.3. This definition is based on that given in [Car62], which is different from that of [GAGA]. The definition by Serre corresponds to the reduced analytic spaces under this definition.

Remark 1.4. If X is smooth, then it is simply a complex manifold. Let $f : X \to Y$ be a morphism of locally ringed space. If $x \in X$ and $f(x) \in Y$ are smooth points, then f is locally a holomorphic map in a neighbourhood of x. This is because if we take the germs of the projection maps $(U, \mathrm{pr}_i) \in \mathcal{H}_{Y,f(x)}$, we get that $f^\sharp(U, \mathrm{pr}_i) = (V, \mathrm{pr}_i \circ f) \in \mathcal{H}_{X,x}$ are holomorphic. Hence, we can view the theory of analytic spaces as a generalisation of the theory of complex manifolds.

We note that if $X = \mathbb{C}^n$, then for any $x \in X$, $\mathcal{H}_{X,x} \cong \mathbb{C}\{x_1, \ldots, x_n\}$ is a noetherian local ring. In general, $\mathcal{H}_{X,x} \cong \mathbb{C}\{x_1, \ldots, x_n\}/I$ where I is the ideal generated by a finite set of holomorphic functions $f_1, \ldots, f_k$. Hence, $\mathcal{H}_X$ is a locally noetherian sheaf of rings over $\mathbb{C}$.

Remark 1.5. Indeed, it is possible to show that $\mathcal{H}_{X,x}$ is isomorphic to either $\mathbb{C}\{x_1, \ldots, x_m\}$ for some m or $\mathbb{C}\{x_1, \ldots, x_m\}/I$ where $I \subset \mathfrak{m}^2$. We say that the point x is **smooth** in the former case and that the point x is an **analytic singularity** in the latter.

We conclude with some commutative algebra results central to the GAGA theorems.

Let $\mathcal{O}_X \subset \mathcal{H}_X$ be the subsheaf of regular functions on X.

We present the following propositions by Serre [GAGA] without proof.

Proposition 1.6. *Let $x \in X$, then $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{H}}_{X,x}$.*

Sketch of proof. The proposition is clear in the case where $X = \mathbb{C}^n$. In general, locally around x, there exists an open subset $W \subset X$ such that W is analytic in an open set $U \subset \mathbb{C}^n$. We can then define an ideal sheaf $\mathcal{I}(X) \subset \mathcal{O}_U$ with support W. Let $I = \mathcal{I}(X)_x$. Then it is clear that $\mathcal{O}_{X,x} = \mathcal{O}_{U,x}/I$ and $\mathcal{H}_{X,x} = \mathcal{H}_{U,x}/I \cdot \mathcal{H}_{U,x}$. □

Definition 1.7. Let A be a subring of B. The pair (A, B) is **flat** if B/A is a flat A-module.

Proposition 1.8. *Let A be a subring of B. The following are equivalent:*

(i) *(A, B) is a flat pair.*

(ii) *B is flat over A and for all A-modules (of finite type) E, the homomorphism $E \to E \otimes_A B$ is injective.*

(iii) *B is flat over A and for all ideals $\mathfrak{a} \subset A$, $\mathfrak{a}B \cap A = \mathfrak{a}$.*

Proposition 1.9. *Let $x \in X$, then $(\mathcal{O}_{X,x}, \mathcal{H}_{X,x})$ is a flat pair.*

1.2 Analytic space associated to a $\mathbb{C}$-scheme locally of finite type

Now, we describe how we can associate to each $\mathbb{C}$-scheme locally of finite type an analytic space. We start with an explicit construction.

Let X be a $\mathbb{C}$-scheme and $X(\mathbb{C})$ be the set of $\mathbb{C}$-rational points on X. Then $(X(\mathbb{C}), \mathcal{O}_X|_{X(\mathbb{C})})$ is a locally ringed space. First suppose $X = \mathrm{Spec}\ (\mathbb{C}[x_1, \ldots, x_n]/I)$ is affine, then we can endow $X(\mathbb{C})$ with a finer topology induced as a subspace topology of $\mathbb{C}^n$ for some n. Call this space $X^{\mathrm{an}} \subset \mathbb{C}^n$. Define $\mathcal{H}_{X^{\mathrm{an}}}$ to be the sheaf generated on each stalk by $\mathcal{H}_x/I \cdot \mathcal{H}_x$.

In general, by taking an open affine cover of X, we can glue together the affine analytifications to obtain an analytic space $(X^{\mathrm{an}}, \mathcal{H}_{X^{\mathrm{an}}})$.

There is a natural morphism of locally ringed spaces

$$\phi : (X^{\mathrm{an}}, \mathcal{H}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$$

whose topological image is the set $X(\mathbb{C}) \subset X$ and whose underlying map of sheaves $\phi^*\mathcal{O}_X \to \mathcal{H}_{X^{\mathrm{an}}}$ sends a regular function f on $U \subset X$ to a regular function $f \circ \phi$ on $U^{\mathrm{an}} \subset X^{\mathrm{an}}$. ϕ induces a bijection between the sets X^{an} and $X(\mathbb{C})$.

Proposition 1.10. *For any $x \in X^{\mathrm{an}}$, the morphism of local rings*

$$\phi_x : \mathcal{O}_{X,\phi(x)} \to \mathcal{H}_{X^{\mathrm{an}},x}$$

induces an isomorphism

$$\hat{\phi}_x : \hat{\mathcal{O}}_{X,\phi(x)} \to \hat{\mathcal{H}}_{X^{\mathrm{an}},x}.$$

Proof. This is an immediate consequence of the definition of X^{an} and Prop. 1.6. □

Remark 1.11. This definition of the analytification of a $\mathbb{C}$-scheme is once again slightly different from the more geometric definition given by Serre. However, [GAGA, Prop. 4] shows that the two definitions are equivalent.

It is easy to show that this construction of Φ is universal in the following sense. Let X be a $\mathbb{C}$-scheme. Consider the functor

$$\Phi_X : \mathbf{AnSp} \to \mathbf{Sets} : \mathcal{X} \mapsto \mathrm{Hom}_{\mathbb{C}}(\mathcal{X}, X)$$

which sends an analytic space $\mathcal{X}$ to the set of morphisms of locally ringed spaces $\mathrm{Hom}_{\mathbb{C}}(\mathcal{X}, X)$.

Theorem 1.12. *The functor Φ_X is representable by the analytic space X^{an} and the equivalence $\mathrm{Hom}(-, X^{\mathrm{an}}) \cong \Phi_X$ is induced by pre-composition with ϕ.*

Sketch of proof. We carry out a series of reductions.

1. It suffices to prove the theorem for X affine since in general, the analytification is defined to be the glueing of the analytifications over an affine covering.

2a. Let $Y \hookrightarrow X$ be an open immersion. Then, the theorem is true for Y if it is true for X since $Y^{\mathrm{an}} \cong X^{\mathrm{an}} \times_X Y$ as X^{an} and Y^{an} are locally isomorphic.

2b. Let $Y \hookrightarrow X$ be a closed immersion. Then, the theorem is true for Y if it is true for X. This is because locally if $\mathcal{O}_{Y,x} = \mathcal{O}_{X,x}/I$, then $\mathcal{H}_{Y^{\mathrm{an}},x} = \mathcal{H}_{X^{\mathrm{an}},x}/I \cdot \mathcal{H}_{X^{\mathrm{an}},x}$.

   Hence, it suffices to prove the theorem for $X = \mathbb{A}^n_{\mathbb{C}}$.

3. It is easy to show that $(X \times Y)^{\mathrm{an}} = X^{\mathrm{an}} \times Y^{\mathrm{an}}$, so it suffices to prove the theorem for the affine line $X = \mathbb{A}^1_{\mathbb{C}}$.

We can prove the following natural isomorphisms for all analytical spaces $\mathcal{X}$

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{X}, \mathbb{A}^1_{\mathbb{C}}) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x], \Gamma(\mathcal{X}, \mathcal{H}_{\mathcal{X}})) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\{x\}, \Gamma(\mathcal{X}, \mathcal{H}_{\mathcal{X}})) \cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{X}, (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}),$$

and hence the theorem is proven. □

An immediate consequence of the theorem is that any morphism of schemes $f : X \to Y$ lifts to a unique morphism of locally ringed spaces $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ such that the following diagram commutes:

$$\begin{tikzcd} X^{\mathrm{an}} \arrow[r, "\phi_X"] \arrow[d, "f^{\mathrm{an}}"'] & X \arrow[d, "f"] \\ Y^{\mathrm{an}} \arrow[r, "\phi_Y"] & Y \end{tikzcd}$$

We also obtain that $(X \times_Z Y)^{\mathrm{an}} \cong X^{\mathrm{an}} \times_{Z^{\mathrm{an}}} Y^{\mathrm{an}}$.

2 Comparison of properties of $\mathbb{C}$-schemes and analytic spaces

Definition 2.1. Let R be a local ring. The **depth** of R is the maximal length of a regular sequence in R. It is clear that $\dim R \geq \operatorname{depth} R$. We say that R is **Cohen-Macaulay** if $\dim R = \operatorname{depth} R$. We say that R is (S_n) if $\operatorname{depth} R \geq \min\{\dim R, n\}$.

Let $(X, \mathcal{O}_X)$ be a locally ringed space. We say that $(X, \mathcal{O}_X)$ is **Cohen-Macaulay** ((S_n), respectively) if for each $x \in X$, $\mathcal{O}_{X,x}$ is Cohen-Macaulay or ((S_n),respectively). We say that $(X, \mathcal{O}_X)$ is (R_n) if all $x \in X$ of codimension $\leq n$ are regular.

For simplicity, from now on, we will denote the $\mathbb{C}$-scheme $(X, \mathcal{O}_X)$ by X and the analytic space $(X^{\mathrm{an}}, \mathcal{H}_{X^{\mathrm{an}}})$ by X^{an}.

Proposition 2.2. *Let X be a $\mathbb{C}$-scheme locally of finite type. Let P be one of the following properties:*

(i) *non-empty;*

(ii) *Cohen-Macaulay;*

(iii) (S_n)*;*

(iv) *regular;*

(v) (R_n)*;*

(vi) *normal;*

(vii) *reduced;*

(viii) *of dimension n;*

(ix) *discrete.*

Then, X has property P if and only if X^{an} has property P.

Sketch of proof. Property (i) follows from the fact that there is a bijection between the underlying sets X^{an} and $X(\mathbb{C})$.

Properties (ii)-(vii) are defined locally on each point of X and X^{an}. X is an excellent scheme (since it is a scheme locally of finite type over a field), so it will satisfy each property P ((ii)-(vii)) on an open subset of X [EGA IV, Prop. 7.8.6(iii)]. Hence, it suffices to check that property P holds for each point $x \in X(\mathbb{C})$. For any $x \in X^{\mathrm{an}}$, $\mathcal{H}_{X^{\mathrm{an}},x}$ and $\mathcal{O}_{X,\phi(x)}$ are excellent local rings, and so are their completions. Property P is preserved under completion of excellent local rings, and since ϕ_x induces an isomorphism of the completions (Prop. 1.6), we have that X satisfies property P if and only if X^{an} satisfies property P.

Similarly, for property (viii), we have that the dimension of a local ring is equal to the dimension of its completion, so

$$\dim X = \sup_{x \in X} \dim \mathcal{O}_{X,x} = \sup_{x \in X(\mathbb{C})} \dim \mathcal{O}_{X,x} = \sup_{x \in X^{\mathrm{an}}} \dim \mathcal{H}_{X^{\mathrm{an}},x} = \dim X^{\mathrm{an}}.$$

Property (ix) is equivalent to property (viii) for $n = 0$. □

The morphism $\phi : X^{\mathrm{an}} \to X$ is also well-behaved on subsets of X.

Proposition 2.3. *Let X be a $\mathbb{C}$-scheme locally of finite type and $T \subset X$ a constructible subset. Then, $\phi^{-1}(\overline{T}) = \overline{\phi^{-1}(T)}$. In particular, T is closed (open, dense, respectively) iff and only if $\phi^{-1}(T)$ is closed (open, dense, respectively).*

Proof. It is clear that $\overline{\phi^{-1}(T)} \subset \phi^{-1}(\overline{T})$. Now we may assume $X = \overline{T}$, so $\phi^{-1}(X) = X^{\mathrm{an}}$ and T is open and dense in X since T is constructible. Let Z be the reduced closed subscheme with underlying topological space $X - T$ and so Z^{an} has underlying topological space $X^{\mathrm{an}} - \phi^{-1}(T)$. Let $x \in Z^{\mathrm{an}}$, so $\phi(x) \in Z \subset \overline{T}$. Hence, if $\mathcal{O}_{Z,\phi(x)} \cong \mathbb{C}[x_1, \ldots, x_n]/I$ and $\mathcal{O}_{X,\phi(x)} \cong \mathbb{C}[x_1, \ldots, x_n]/J$, then $\sqrt{J} \subsetneq \sqrt{I}$. However, x is not in the closure of $\phi^{-1}(T)$ if and only if there exists an open set $U \subset X^{\mathrm{an}}$ containing x if and only if $\sqrt{I \cdot \mathbb{C}\{x_1, \ldots, x_n\}} = \sqrt{J \cdot \mathbb{C}\{x_1, \ldots, x_n\}}$ which is clearly false.

Now, if $\phi^{-1}(T)$ is closed, then $\phi^{-1}(\overline{T}) = \phi^{-1}(T)$, that is, T and $\overline{T}$ have the same $\mathbb{C}$-rational points. Since T is constructible, we can write $T = \bigcup_{i=1}^{n} (U_i \cap Z_i)$ for some open sets U_i and closed sets Z_i. If $T \neq \overline{T}$, then there exists some i such that $U_i \cap Z_i \neq \overline{U}_i \cap Z_i$. If $x \in (\overline{U}_i - U_i) \cap Z_i$, then so does any $\mathbb{C}$-rational point $z \in \overline{\{x\}}$. Hence, we get a contradiction and we must have $T = \overline{T}$.

$\phi^{-1}(T)$ is open iff $\phi^{-1}(X - T)$ is closed iff $X - T$ is closed iff T is open.

If T is dense in X, then $\overline{\phi^{-1}(T)} = \phi^{-1}(\overline{T}) = X^{\mathrm{an}}$. Conversely, if $\phi^{-1}(T)$ is dense in X^{an}, then $X^{\mathrm{an}} = \phi^{-1}(\overline{T})$ so $\overline{T}$ contains all the $\mathbb{C}$-rational points of X. Since T is constructible, so $\phi^{-1}(T)$ is dense and constructible, hence open, and so is T. Suppose $x \notin \overline{T}$, hence there is an open neighbourhood V of x such that $V \cap T = \emptyset$. Let $U = \mathrm{Spec}\ (\mathbb{C}[x_1, \dots, x_n]/I) \subset X$ be an open affine subscheme containing x. Then $V \cap U$ is open in U, and hence contains some $\mathbb{C}$-rational points. Thus, $\phi^{-1}(V)$ is open in X^{an} and is disjoint from $\phi^{-1}(T)$, contradicting the hypothesis that $\phi^{-1}(T)$ is dense in X^{an}. □

Definition 2.4. Let $\mathcal{X}$ be an analytic space. X is **irreducible** if and only if it cannot be decomposed into a finite number of closed sub-analytic spaces $\mathcal{X}_i \neq \mathcal{X}$.

Proposition 2.5. *Let X be a $\mathbb{C}$-scheme locally of finite type. Then X is connected (irreducible, respectively) if and only if X^{an} is connected (irreducible, respectively).*

Sketch of proof. If X^{an} is connected or irreducible, then so is $X(\mathbb{C})$ and hence X.

To prove the converse, we can carry out some reduction steps: suppose X is connected/irreducible,

1. We may assume X is irreducible. If X is connected but not irreducible, then for any two points $x, y \in X^{\mathrm{an}}$, we can find a sequence $X_1, \dots, X_n$ of irreducible components of X such that $x \in X_1$, $y \in X_n$ and $X_i \cap X_{i+1} \neq \emptyset$ for all $i < n$. Suppose each X_i^{an} is irreducible, and hence connected, then $(X_i \cap X_{i+1})^{\mathrm{an}}$ is non-empty for each $i < n$, so there exists a path in X^{an} from x to y. Therefore X^{an} is connected.
2. We may assume that $X = \mathrm{Spec}\ A$ is affine. Indeed, if X is irreducible, it has a cover by dense open affine schemes U_i. If each U_i^{an} is irreducible, then so is X^{an} since each U_i^{an} is also open and dense.
3. Since we are only concerned about the topology of X, we may assume X is a reduced scheme, hence integral.
4. It suffices to show that X^{an} is connected. Then, since all the local rings of X^{an} are integral, X^{an} is irreducible.

To complete the proof, we let $i : X \to P$ be a compactification of X, for example by taking the closure of X in the embedding $X \hookrightarrow \mathbb{A}^n_{\mathbb{C}} \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$. P is a integral projective scheme so P^{an} is connected iff P is by Cor. 4.4.

It remains to show that X^{an} is connected given that P^{an} is connected. By the previous proposition, X^{an} is dense in P^{an} so $Y^{\mathrm{an}} = P^{\mathrm{an}} - X^{\mathrm{an}}$ is an analytic space with complex codimension ≥ 1. The set of singularities Z of Y^{an} has complex codimension 2, so by [Ser66, Prop. 4], $H^0(P^{\mathrm{an}}) = H^0(P^{\mathrm{an}} - Z)$ and $H^0(Y^{\mathrm{an}}) = H^0(Y^{\mathrm{an}} - Z)$. For each $x \in Y^{\mathrm{an}} - Z$, by the implicit function theorem, there exists a neighbourhood U of X^{an} where $U \cong \mathbb{C}^n - \mathbb{C}^m$ for some $m < n$. Hence U is connected and so $H^0(P^{\mathrm{an}} - Z) = H^0(X^{\mathrm{an}})$. □

Corollary 2.6. *Let X be a $\mathbb{C}$-scheme locally of finite type. Then X is integral if and only if X^{an} is integral.*

Proof. A scheme or analytic space is integral iff it is reduced and each component is irreducible. □

3 Comparison of properties of morphisms

Definition 3.1. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. f is **smooth** (**normal**, **reduced**, respectively) if f is flat and for all $x \in X$ and $y = f(x) \in Y$, the geometric fibres $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \overline{k(y)}$ of f are regular (normal, reduced, respectively).

Proposition 3.2. *Let $f : X \to Y$ be a morphism of $\mathbb{C}$-schemes locally of finite type and $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ the induced map of analytic spaces. Let P be one of the following properties:*

(i) flat;

(ii) unramified;

(iii) étale;

(iv) smooth;

(v) normal;

(vi) reduced;

(vii) injective;

(viii) an open immersion;

(ix) an isomorphism;

(x) a monomorphism;

Then, f satisfies property P if and only if f^{an} satisfies property P.

If we further assume that f is of finite type, let P be one of the following properties:

(xi) surjective;

(xii) dominant;

(xiii) a closed immersion;

(xiv) an immersion;

(xv) projective.

(xvi) proper;

(xvii) finite.

Then, f satisfies property P if and only if f^{an} satisfies property P.

Sketch of proof Note that (i)-(vi) are defined locally, and by [EGA IV], if f verifies these properties on a dense open subset of X, it verifies them on all of X. Hence it suffices to check them on points in $X(\mathbb{C})$.

Let $x \in X^{\mathrm{an}}$ and $y = f(x) \in Y^{\mathrm{an}}$. We have a cartesian diagram

$$\begin{tikzcd} \mathcal{O}_{Y,y} \arrow[r, "f_x"] \arrow[d] & \mathcal{O}_{X,x} \arrow[d] \\ \mathcal{H}_{Y^{\mathrm{an}},y} \arrow[r, "f_x^{\mathrm{an}}"] & \mathcal{H}_{X^{\mathrm{an}},x} \end{tikzcd}$$

Hence, for $x \in X^{\mathrm{an}}$, f_x satisfies (i) or (ii) iff f_x^{an} satisfies (i) or (ii) respectively. (i) and (ii) imply (iii).

(iv)-(vi) follow from (i) and (iv), (vi) and (vii) respectively of Prop. 2.2 applied to the locally ringed spaces $X \times_Y \operatorname{Spec} k(y)$ and $X^{\mathrm{an}} \times_{Y^{\mathrm{an}}} (\operatorname{Spec} k(y))^{\mathrm{an}} \cong (X \times_Y \operatorname{Spec} k(y))^{\mathrm{an}}$.

(vii). It is clear that f^{an} is injective iff $f|_{X(\mathbb{C})}$ is injective. It remains to show that f is injective if $f|_{X(\mathbb{C})}$ is. Suppose there exist distinct $x, x' \in X$ such that $y = f(x) = f(x')$, then f maps $\overline{\{x\}}$ and $\overline{\{x'\}}$ to dense open subsets of $\overline{\{y\}}$ (since the image of a constructible set is constructible and any dense constructible subset of an irreducible closed set is open in that set). We have

$$\dim(f(\overline{\{x\}}) \cap f(\overline{\{x'\}})) = \dim \overline{\{y\}} \qquad \text{and} \qquad \dim f(\overline{\{x\}} \cap \overline{\{x'\}}) < \dim \overline{\{y\}}.$$

Hence, there exists some dense open subset $U \subset \overline{\{y\}}$ disjoint from $f(\overline{\{x\}} \cap \overline{\{x'\}})$ but have non-empty intersection with $f(\overline{\{x\}})$ and $f(\overline{\{x'\}})$. Thus, there exist points $t \in \overline{\{x\}} \cap X(\mathbb{C})$ and $t' \in \overline{\{x'\}} \cap X(\mathbb{C})$ such that $f(t) = f(t')$.

(viii) An open immersion is an injective étale morphism, so (viii) follows from (iii) and (vii). An isomorphism is a surjective open immersion, so (ix) follows from (viii) and (xi) (since an open immersion is of finite type). f is a monomorphism iff $\Delta : X \to X \times_Y X$ is an isomorphism, so (x) follows from (ix).

(xi) and (xii) follows from Prop. 2.3. f (or f^{an}) is dominant iff $f(X)$ (or $f^{\mathrm{an}}(X^{\mathrm{an}}) = f(X)^{\mathrm{an}}$) is dense in Y (or Y^{an}). It is surjective iff it is dense and closed.

(xiii) follows from (x) and (xvi) since a closed immersion is a proper monomorphism.

(xiv) Suppose f is an immersion, then it factors as the composition of an open immersion followed by a closed immersion. Hence, f^{an} is also an immersion.

Conversely, suppse f^{an} is an immersion. Let $T = f(X)$ be the image of X and f factorises as the composition

$$X \xrightarrow{i} \overline{T} \xrightarrow{j} Y$$

where j is a closed immersion. Since $f(X)$ is constructible, T is open in $\overline{T}$, so we can again factor i as

$$X \to i(X) = T \to \overline{T},$$

where the second map is an open immersion. Taking the analytification, we get

$$X^{\mathrm{an}} \to i(X)^{\mathrm{an}} \cong i^{\mathrm{an}}(X^{\mathrm{an}}) \to \overline{T}^{\mathrm{an}},$$

where the second map is an open immersion by (ix), while the first is an isomorphism since the composition is an open immersion (as it is a factor of f^{an}). By (x), we thus have $X \cong i(X)$ and so i is an open immersion.

(xv) A projective map is a composition of a closed immersion with a standard projection $\mathbb{P}^r_Y \to Y$, hence the result follows from (xiii).

(xvi) By assumption, f is of finite type and separated, hence so is f^{an}. Hence, it suffices to show that f is universally closed iff f^{an} is. Indeed, it suffices to show that f is closed iff f^{an} is. Suppose f is proper, then for any extension under base change $h : X \times_Y Y' \to Y'$, h is proper and hence closed, so h^{an} is closed, and hence f is universally closed. The converse argument is similar.

Now, suppose f^{an} is closed and let $T \subset X$ be a closed subset. $f(T)^{\mathrm{an}} = f^{\mathrm{an}}(T^{\mathrm{an}})$ is closed by hypothesis, so $f(T)$ is closed by Prop. 2.3.

Conversely, suppose f is proper. Then, by Chow's lemma, there exists a scheme X' projective over Y with a surjective projective morphism $g : X' \to X$ such that the following diagram commutes:

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[dr] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

By (xi) and (xv), X'^{an} is projective over Y^{an} and g^{an} is projective and surjective. Hence, f^{an} is closed.

(xvii) A finite morphism is a proper morphism with finite fibres. If f has finite fibres, then clearly so does f^{an}. Conversely, if f^{an} has finite fibres, then for all $y \in Y(\mathbb{C})$, $f^{-1}(y)$ has finite fibres. The set of $y \in Y$ with finite fibre is constructible and includes all of $Y(\mathbb{C})$, hence it is equal to Y. Thus, f has finite fibre. The result then follows from (xvi). $\square$

Remark 3.3. Note that the condition that f is of finite type is necessary for (xi)-(xvii). For example, the inclusion

$$f : \coprod_{i \in \mathbb{Z}} \operatorname{Spec} \mathbb{C} \to \mathbb{A}^1_{\mathbb{C}}$$

send one copy of Spec $\mathbb{C}$ to each integer is not a closed immersion since the image is not closed in $\mathbb{A}^1_{\mathbb{C}}$ but f^{an} is a closed immersion since the image is the zeros of the holomorphic function $\sin z$.

4 The GAGA theorems

The main result of GAGA proven by Serre [GAGA] is the following statement: Let X be a projective $\mathbb{C}$-scheme locally of finite type; there is an equivalence between the category $\mathbf{Coh}_X$ of coherent sheaves on $(X, \mathcal{O}_X)$ and the category $\mathbf{Coh}_{X^{\mathrm{an}}}$ of coherent sheaves on $(X^{\mathrm{an}}, \mathcal{H}_{X^{\mathrm{an}}})$. Grothendieck [SGA I, Éxpose XII] generalised this result to proper morphisms of $\mathbb{C}$-schemes locally of finite type.

First, we define the functor $\mathbf{Coh}_X \to \mathbf{Coh}_{X^{\mathrm{an}}}$. Let $\phi : X^{\mathrm{an}} \to X$ be the canonical morphism. Given a coherent sheaf $F \in \mathbf{Coh}_X$, consider the sheaf $\phi^* F = \phi^{-1} F \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \mathcal{H}_{X^{\mathrm{an}}}$. Since ϕ is a flat morphism of locally ringed spaces, ϕ^* sends coherent sheaves to coherent sheaves. We thus have a well-defined functor

$$\begin{aligned} \phi^* \quad : \quad & \mathbf{Coh}_X \to \mathbf{Coh}_{X^{\mathrm{an}}} \\ & F \mapsto F^{\mathrm{an}} = \phi^* F \end{aligned}$$

Note that if $F = \mathcal{O}_X$, then $\phi^* F = \mathcal{H}_{X^{\mathrm{an}}}$.

Proposition 4.1. *The functor $F \mapsto F^{\mathrm{an}}$ is exact, faithful and conservative (isomorphism-reflecting).*

Sketch of proof. Exactness follows from the fact that ϕ^{-1} is exact (since it has both left and right adjoints) and $\mathcal{H}_{X^{\mathrm{an}}}$ is flat over $\mathcal{O}_{X^{\mathrm{an}}}$ (Prop. 1.9).

Since $\mathcal{O}_{X^{\mathrm{an}}} \to \mathcal{H}_{X^{\mathrm{an}}}$ is faithfully flat, for each $x \in X^{\mathrm{an}}$, $F_x \otimes_{\mathcal{O}_{X^{\mathrm{an}},x}} \mathcal{H}_{X^{\mathrm{an}},x} = 0$ iff $F_x = 0$. Since $X(\mathbb{C})$ is dense in X, we conclude that $F = 0$ iff $F_x = 0$ for all $x \in X(\mathbb{C})$.

Conservativeness follows from faithfulness and flatness. $\square$

ϕ^* has a left adjoint ϕ_*, and hence there is a canonical natural morphism $F \mapsto \phi_* \phi^* F = \phi_* F^{\mathrm{an}}$ for any $F \in \mathbf{Mod}(\mathcal{O}_X)$.

Let $f : X \to Y$ be a morphism of $\mathbb{C}$-schemes locally of finite type, $\phi : X^{\mathrm{an}} \to X$ and $\psi : Y^{\mathrm{an}} \to Y$ be the canonical morphisms. The pushforward $f_* : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_Y)$ is left exact, since it has a left adjoint. so we can consider the right derived functors $R^\bullet f_*$. For any $F \in \mathbf{Mod}(\mathcal{O}_X)$, there are functorial maps

$$R^\bullet f_* F \to R^\bullet f_*(\phi_* F^{\mathrm{an}}) \to R^\bullet (f \circ \phi)_* F^{\mathrm{an}} = R^\bullet (\psi \circ f^{\mathrm{an}})_* F^{\mathrm{an}} \to \psi_*(R^\bullet f_*^{\mathrm{an}} F^{\mathrm{an}})$$

where the last morphism is induced as follows: since (ψ^*, ψ_*) is an adjoint pair, there is a canonical natural transformation $\psi^* \psi_* \to \mathrm{id}_{\mathbf{Mod}(\mathcal{H}_{Y^{\mathrm{an}}})}$. This induces a morphism of complexes

$$\psi^* \psi_* f_*^{\mathrm{an}} I^\bullet \to f_*^{\mathrm{an}} I^\bullet$$

where $I^\bullet$ is an injective resolution of F^{an}. This induces a natural morphism of right derived functors

$$\psi^* R^\bullet \psi_* f_* F^{\mathrm{an}} \cong R^\bullet \psi^* \psi_* f_* F^{\mathrm{an}} \to R^\bullet f_* F^{\mathrm{an}}$$

where the first isomorphism is true because ψ^* is exact. The required morphism then follows by taking the adjoint morphism.

The sequence of morphisms of $\mathcal{O}_Y$-modules induces a morphism

$$\theta^\bullet : (R^\bullet f_* F)^{\mathrm{an}} \to R^\bullet f_*^{\mathrm{an}} (F^{\mathrm{an}})$$

of $\mathcal{H}_{Y^{\mathrm{an}}}$-modules.

It is not difficult to give an explicit description of the morphisms in terms of sections of sheaves in an injective resolution of F, but it is not very instructive. Instead we consider the Čech cohomology.

Now assume that $f : X \to Y$ is a proper morphism, so f_* sends coherent $\mathcal{O}_X$-modules to coherent $\mathcal{O}_Y$ modules [EGA III, Thm. 3.2.1]. If F is a quasi-coherent sheaf over a $\mathbb{C}$-scheme X locally of finite type, there is an isomorphism $\mathcal{O}_Y$-modules, for each p,

$$R^p f_* F \cong \check{H}^p(X, F, f_*)) = \varinjlim_{\mathcal{U}} H^p(\mathcal{C}^\bullet(\mathcal{U}, F, f_*))$$

where $\mathcal{U} = \{U_i\}_{i \in I}$ is some open cover of X and the colimit is taken over refinement of covers, and $\mathcal{C}^\bullet(\mathcal{U}, F, f_*)$ is the Čech complex with terms

$$\mathcal{C}^q(\mathcal{U}, F, f_*) = \bigoplus_{|J| = q, J \subset I} (f|_{U_J})_* F|_{U_J}$$

where $U_J = \bigcap_{j \in J} U_j$ and the differentials are the standard Čech differentials.

A fundamental result in Čech cohomology is that if there exists an open covering $\mathcal{U}$ of X (e.g. open Y-affine covering of X, i.e. each element of the cover is an algebraic subset of $\mathbb{A}^r_Y$) such that $H^p(\mathcal{C}^\bullet(\mathcal{U}, I, f_*)) = 0$ for all $p > 0$ and all injective $\mathcal{O}_X$-modules I, then

$$R^p f_* F \cong H^p(\mathcal{C}^\bullet(\mathcal{U}, F, f_*)) \cong \check{H}^p(X, F, f_*))$$

For morphisms of analytic spaces, we have a similar relationship between Čech cohomology and sheaf cohomology. In fact, as the topology is finer (it is paracompact and Hausdorff), all sheaves, and not just coherent sheaves, can be "resolved" using Čech cohomology. Hence, if $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ is a proper morphism of analytic spaces and $\mathcal{F} \in \mathbf{Mod}(\mathcal{H}_\mathcal{X})$, then

$$R^p \tilde{f}_* \mathcal{F} \cong \check{H}^p(\mathcal{X}, \mathcal{F}, \tilde{f}_*)) = \varinjlim_{\mathcal{U}} H^p(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}, \tilde{f}_*)).$$

Indeed, for any open $\mathcal{Y}$-affine covering $\mathcal{U}$ of $\mathcal{X}$, we have

$$R^p \tilde{f}_* \mathcal{F} \cong H^p(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}, \tilde{f}_*)).$$

Hence, the morphism $\theta^\bullet$ embeds the set of sections of F on U_J ($|J| = p+1$) into the set of sections of F^{an} on U_J^{an} and the constructions above show that the embedding commutes with the derived functors $R^\bullet f_*$ and $R^\bullet f_*^{\mathrm{an}}$. The next theorem shows that the map is indeed an isomorphism.

Theorem 4.2. *Let $f : X \to Y$ be a proper morphism of $\mathbb{C}$-schemes locally of finite type, and $F \in \mathbf{Coh}_X$. Then, for any $p \geq 0$, the morphism*

$$\theta^p : (R^p f_* F)^{\mathrm{an}} \to R^p f_*^{\mathrm{an}}(F^{\mathrm{an}})$$

is an isomorphism.

Remark 4.3. The case where $Y = \operatorname{Spec} \mathbb{C}$ and f is projective was proven by Serre. Grothendieck generalised it to proper morphisms.

Sketch of proof of theorem. Case 1: f is projective.

1. First we assume $f : X = \mathbb{P}^r_Y \to Y$ is the standard projection and $F = \mathcal{O}_X$. Then using the standard open covering $\mathcal{U} = \{U_i\}_{i=0,\dots,r}$ of X, we can show that $R^0 f_* F = H^0(Y, \mathcal{O}_Y)$ and $R^p f_* F = 0$ for all $p > 0$. Similarly, we can show that $R^0 f_*^{\mathrm{an}} F^{\mathrm{an}} = H^0(Y^{\mathrm{an}}, \mathcal{H}_{Y^{\mathrm{an}}})$ and $R^p f_*^{\mathrm{an}} F^{\mathrm{an}} = 0$ for all $p > 0$.
2. Next, by inducting on the dimension of X as a Y-projective space and considering the exact sequences of sheaves
$$0 \to \mathcal{O}(n-1) \to \mathcal{O}(n) \to \mathcal{O}_E(n) \to 0$$
where E is some hyperplane in X, we can verify the theorem for all standard projections $f : X = \mathbb{P}^r_Y \to Y$ and twisted sheaves $F = \mathcal{O}_X(n)$.
3. In general, let $i : X \to \mathbb{P}^r_Y$ be a closed immersion, then for any $F \in \mathbf{Coh}_X$, we can show that $(i_* F)^{\mathrm{an}} \cong i_*^{\mathrm{an}} F^{\mathrm{an}}$ by checking on stalks. Furthermore, any coherent sheaf R on $\mathbb{P}^r_Y$ can be written as the cokernel of the direct sum of some $\mathcal{O}(n)$. Hence, we get an exact sequence
$$0 \to L \to \bigoplus \mathcal{O}(n_i) \to R \to 0.$$
By inducting on the rank of R and using the long exact sequence associated to this short exact sequence, we obtain the necessary results.

Case 2: f is proper. The "lemme de dévissage" [EGA III, Thm. 3.1.2] shows that it suffices to verify the theorem for a set of sheaves $\mathbf{F} \subset \mathbf{Coh}_X$ such that for each $x \in X$, there exists $F \in \mathbf{F}$ such that $F_x \neq 0$. Chow's lemma states that given any proper morphism $f : X \to Y$, there exists a surjective projective morphism $g : X' \to X$ such that $g \circ f$ is projective as well. It then suffices to choose $F = g_*(\mathcal{O}_{X'}(n))$ for some suitable n. The verification of the theorem is an exercise in sheaf cohomology. ☐

Corollary 4.4. *Let X be a proper $\mathbb{C}$-scheme and $F \in \mathbf{Coh}_X$. Then, for all $p > 0$, there is an isomorphism*

$$H^p(X, F) \to H^p(X^{\mathrm{an}}, F^{\mathrm{an}}).$$

Proof. Take $Y = \text{Spec}\,\mathbb{C}$ and $f : X \to Y$ to be the canonical map and apply Thm. 4.2. $\square$

Remark 4.5. It is clear that the GAGA theorems do not hold without the assumption of properness. For example, on the affine line, $\mathbb{A}^1_{\mathbb{C}}$, there are many analytic functions (eg. $\sin z$) that are not regular. Hence, $H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) \ncong H^0(\mathbb{C}^1, \mathcal{H}_{\mathbb{C}^1})$. The theorem is a statement that all holomorphic functions on compact spaces are regular.

The main theorem in GAGA is the following:

Theorem 4.6. *Let X be a proper $\mathbb{C}$-scheme and $\phi : X^{\text{an}} \to X$ be the canonical morphism. Then the functor $\phi^* : \mathbf{Coh}_X \to \mathbf{Coh}_{X^{\text{an}}}$ sending F to F^{an} is an equivalence of categories.*

Sketch of proof. We will only show that the functor is fully faithful. The proof of essential surjectivity is a rather involved construction which is beyond the scope of these notes.

Let $F, G \in \mathbf{Coh}_X$ and consider the coherent sheaf $\mathbf{Hom}_{\mathcal{O}_X}(F, G)$. By Cor. 4.4, there is an isomorphism

$$\text{Hom}_{\mathcal{O}_X}(F, G) \cong H^0(X, \mathbf{Hom}_{\mathcal{O}_X}(F, G)) \xrightarrow{\sim} H^0(X^{\text{an}}, \mathbf{Hom}_{\mathcal{O}_X}(F, G)^{\text{an}}).$$

The latter sheaf $\mathbf{Hom}_{\mathcal{O}_X}(F, G)^{\text{an}}$ is isomorphic to $\mathbf{Hom}_{\mathcal{H}_{X^{\text{an}}}}(F^{\text{an}}, G^{\text{an}})$ since F is coherent and $\mathcal{H}_{X^{\text{an}}}$ is flat over $\mathcal{O}_{X^{\text{an}}}$. Hence, we have an isomorphism

$$\text{Hom}_{\mathcal{O}_X}(F, G) \xrightarrow{\sim} \text{Hom}_{\mathcal{H}_{X^{\text{an}}}}(F^{\text{an}}, G^{\text{an}}).$$

$\square$

5 An application of the GAGA theorems: Chow's theorem

Proposition 5.1 (Chow's theorem)**.** *Let X be a proper $\mathbb{C}$-scheme. Then any closed analytic subspace $\mathcal{Y}$ of X^{an} is the analytification of some closed subscheme $Y \subset X$, that is to say, $\mathcal{Y}$ is* **algebraic***.*

Proof. Let $i : \mathcal{Y} \to X^{\text{an}}$ be the inclusion map. Then there exists a coherent sheaf $\mathcal{A}_{\mathcal{Y}} \in \mathbf{Coh}_{X^{\text{an}}}$ such that $i_*\mathcal{H}_{\mathcal{Y}} \cong \mathcal{H}_{X^{\text{an}}}/\mathcal{A}_{\mathcal{Y}}$. By Thm. 4.6, there exists $A_Y \in \mathbf{Coh}_X$ such that $\mathcal{A}_{\mathcal{Y}} \cong (A_Y)^{\text{an}}$. The support of A_Y is closed. If we let $Y = \text{Supp}(A_Y)$, we can define an inclusion of spaces $j : Y \to X$ and define the locally ringed subspace $Y = (Y, j^{-1}(\mathcal{O}_X/A_Y))$. Then, it is clear that Y is a closed subscheme of X and easy to check that $Y^{\text{an}} = \mathcal{Y}$. $\square$

Proposition 5.2. *Let X be a $\mathbb{C}$-scheme locally of finite type. Then, all proper analytic subspaces of X^{an} are algebraic.*

Proof. There is an open covering of X by $\mathbb{C}$-schemes of finite type U_i. Let $\mathcal{Z}$ be a proper analytic subspace of X^{an}, then $\mathcal{Z}$ can be covered by finitely many U_i^{an}. Hence, without loss of generality, we may assume X is of finite type.

By the Nagata compactification theorem, there is an open immersion $i : X \to Y$ of X into a proper $\mathbb{C}$-scheme Y. $\mathcal{Z}$ is closed in Y^{an}, hence by Chow's theorem, there exists a closed subscheme $Z \subset Y$ such that $Z^{\text{an}} = \mathcal{Z}$. Then,

$$(Z \times_Y X)^{\text{an}} \cong Z^{\text{an}} \times_{Y^{\text{an}}} X^{\text{an}} \cong Z^{\text{an}} \cong \mathcal{Z}$$

since $\mathcal{Z}$ is proper in X^{an}. Hence, $\mathcal{Z}$ is algebraic. $\square$

To prove the next result, we require a lemma.

Lemma 5.3. *Let S be a $\mathbb{C}$-scheme locally of finite type. Let X be a proper S-scheme and Y an S-scheme locally of finite type. Then there is a bijection of sets*

$$\text{Hom}_S(X, Y) \cong \{Z \underset{\textit{closed}}{\subset} X \times_S Y \mid Z \xrightarrow[\text{pr}_1]{\sim} X\}.$$

Furthermore, if we suppose $X \to S$ is étale, and let $\mathrm{Hom}_{S,ét}(X,Y)$ be the set of étale morphisms from X to Y, then there is a bijection of sets

$$\mathrm{Hom}_{S,ét}(X,Y) \cong \{Z \underset{\textit{open and closed}}{\subset} X \times_S Y \mid Z \xrightarrow[\mathrm{pr}_1]{\sim} X\}.$$

The same results hold for analytic spaces and closed analytic subspaces.

Proof. We define a map $\mathrm{Hom}_S(X,Y) \to \{Z \subset X \times_S Y\}$ as follows. For any S-morphism $f : X \to Y$, the universal property of the fibred product gives a unique morphism $i : X \to X \times_S Y$ in the following diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists ! i"] \arrow[drr, "f"] \arrow[ddr, "\mathrm{id}"'] & & \\ & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & S \end{tikzcd}$$

i is a monomorphism since $\mathrm{id} : X \to X$ is. Since $X \to S$ is proper and $Y \to S$ is separated, $f : X \to Y$ is proper [Har77, Chapter II, Cor. 4.8]. $X \times_S Y \to Y$ is proper since proper morphisms are stable under base change, so i is proper as well and hence i is a closed immersion. The image $Z = i(X)$ is the graph of f in $X \times_S Y$, and is hence isomorphic to X.

Conversely, given a closed subscheme $Z \subset X \times_S Y$ such that $Z \xrightarrow{\mathrm{pr}_1} X$ is an isomorphism, The inverse of the restricted projection map $i : X \to X \times_S Y$ gives a S-morphism $X \xrightarrow{i} X \times_S Y \to Y$. It is easy to check the two maps are inverse to each other.

Now, if we suppose $X \to S$ and $f : X \to Y$ are étale, then so is $X \times_S Y \to Y$ since being étale is stable under base change and hence so is $i : X \to X \times_S Y$. An étale monomorphism is an open immersion, so $i(X)$ is an open and closed subscheme of $X \times_S Y$.

Conversely, we see from the construction of the inverse map that if $Z \subset X \times_S Y$ is an open subscheme, then the inclusion map is étale and so is the composition $X \xrightarrow{i} X \times_S Y \to Y$. □

Proposition 5.4. *Let S be a $\mathbb{C}$-scheme locally of finite type. Let X be a proper S-scheme and Y be a S-scheme locally of finite type. Then the canonical morphism*

$$\mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_{S^{\mathrm{an}}}(X^{\mathrm{an}}, Y^{\mathrm{an}})$$

is an isomorphism.

Proof. By Lemma 5.3, it suffices to show that the functor

$$\begin{array}{ccc} \{Z \underset{\text{closed}}{\subset} X \times_S Y \mid Z \xrightarrow[\mathrm{pr}_1]{\sim} X\} & \longrightarrow & \{\mathcal{Z} \underset{\text{closed}}{\subset} X^{\mathrm{an}} \times_{S^{\mathrm{an}}} Y^{\mathrm{an}} \mid \mathcal{Z} \xrightarrow[\mathrm{pr}_1]{\sim} X^{\mathrm{an}}\} \\ Z & \mapsto & Z^{\mathrm{an}} \end{array}$$

induces an equivalence of categories.

Essential surjectivity follows from Prop. 5.2 and Prop. 3.2. $\mathrm{Hom}_{X \times_S Y}(Z, Z') \neq \emptyset$ iff $Z \subset Z'$ iff $Z^{\mathrm{an}} \subset Z'^{\mathrm{an}}$ iff $\mathrm{Hom}_{X^{\mathrm{an}} \times_{S^{\mathrm{an}}} Y^{\mathrm{an}}}(Z^{\mathrm{an}}, Z'^{\mathrm{an}}) \neq \emptyset$. □

Next, we present another consequence of the GAGA theorems. First, we require a lemma.

Lemma 5.5. *Let X be a $\mathbb{C}$-scheme locally of finite type. Then, the functors*

$$\begin{array}{ccc} \{X' \to X \textit{ finite}\} & \longrightarrow & \mathbf{Coh\text{-}Alg}_X \\ (X' \xrightarrow{f} X) & \mapsto & f_* \mathcal{O}_{X'} \end{array}$$

and

$$\begin{array}{ccc} \{\mathcal{X}' \to X^{\mathrm{an}} \textit{ finite}\} & \longrightarrow & \mathbf{Coh\text{-}Alg}_{X^{\mathrm{an}}} \\ (\mathcal{X}' \xrightarrow{f} X^{\mathrm{an}}) & \mapsto & f_* \mathcal{H}_{\mathcal{X}'} \end{array}$$

give rise to equivalences of categories.

Proof. The proof is left as an easy exercise for the reader. □

Proposition 5.6 (Riemann's existence theorem for proper schemes)**.** *Let X be a proper $\mathbb{C}$-scheme. Then, the functor*

$$\begin{array}{ccc} \{X' \to X \textit{ finite (étale)}\} & \longrightarrow & \{\mathcal{X}' \to X^{\mathrm{an}} \textit{ finite (étale)}\} \\ X' & \mapsto & X'an \end{array}$$

induces an equivalence of categories.

Proof. The case for finite morphisms follow from Lemma 5.5 and Thm. 4.6. The case for finite étale morphisms then follow from Prop. 3.2(iii). □

6 Riemann's existence theorem

Informally, Riemann's existence theorem states that the category $\mathbf{FEt}_X$ of finte étale coverings of a $\mathbb{C}$-scheme X locally of finite type is equivalent to the category $\mathbf{FEt}_{X^{\mathrm{an}}}$ of finite étale coverings of X^{an}.

First, we give the formal definition of a covering.

Definition 6.1. Let X be a $\mathbb{C}$-scheme locally of finite type. A morphism $f : X' \to X$ is a finite covering of X if it is finite and the image of any irreducible component (i.e. maximal irreducible subset) of X' is an irreducible component of X. We similarly define finite coverings of analytic spaces.

Remark 6.2. Any flat finite morphism of schemes $f : X' \to X$ is a flat finite covering. Similarly, any flat finite morphism of analytic spaces $f : \mathcal{X}' \to \mathcal{X}$ is a flat finite covering. In particular, the finite étale coverings are precisely the finite étale morphisms.

Remark 6.3. The finite étale coverings of an integral analytic space corresponds precisely with the topological finite coverings of the space.

For simplicity, if $X' \to X$ is a finite covering of X and $Y \subset X$ is a subscheme, we will often write $X'|_Y$ for the fibred product $X' \times_X Y$.

We can now formally state Riemann's existence theorem.

Theorem 6.4. *Let X be a $\mathbb{C}$-scheme locally of finite type. The functor*

$$\begin{array}{ccc} \Phi : \mathbf{FEt}_X & \longrightarrow & \mathbf{FEt}_{X^{\mathrm{an}}} \\ (X' \xrightarrow{f} X) & \mapsto & (X'^{\mathrm{an}} \xrightarrow{f^{\mathrm{an}}} X^{\mathrm{an}}) \end{array}$$

induces an equivalence between the categories of finite étale coverings of X and X^{an}.

Proof. By Prop. 5.4, the maps

$$\mathrm{Hom}_X(X', X'') \to \mathrm{Hom}_{X^{\mathrm{an}}}(X'^{\mathrm{an}}, X''^{\mathrm{an}})$$

are isomorphisms for all X', X'' finite over X, hence the functor Φ is fully faithful.

To prove that it is essentially surjective, we first show that it suffices to assume X is a regular affine scheme.

Let $\mathcal{X}' \to X^{\mathrm{an}}$ be a finite étale covering. The reduction steps are as follows:

1. We may assume that X is affine connected.

   In general, take an affine open cover $\{U_i = \mathrm{Spec}\ A_i\}$ of X. By hypothesis, for each i, there exists an étale finite cover $U_i' \to U_i$ such that $U_i'^{\mathrm{an}} \cong \mathcal{X}'|_{U_i^{\mathrm{an}}}$. By Prop. 3.2(x) and the glueing of $U_i'^{\mathrm{an}}$ over X^{an}, we get, for any i, j,
   $$U_i'|_{U_i \cap U_j} \cong U_j'|_{U_i \cap U_j}.$$
   Hence, we may glue $\{U_i'\}$ over X to obtain a finite étale cover $X' \to X$, and it is clear that $X'^{\mathrm{an}} \cong \mathcal{X}'$.

2. We may assume X is reduced.

   The canonical closed immersion $X_{\mathrm{red}} \to X$ induces functors

$$\mathbf{FEt}_X \to \mathbf{FEt}_{X_{\mathrm{red}}} \qquad \text{and} \qquad \mathbf{FEt}_{X^{\mathrm{an}}} \to \mathbf{FEt}_{X^{\mathrm{an}}_{\mathrm{red}}}.$$

   Using the universal property of $X_{\mathrm{red}} \to X$, it is easy to show that the functors above induce equivalences of categories (an exercise for the readers). Hence, the commutative square

$$\begin{tikzcd} \mathbf{FEt}_X \arrow[r] \arrow[d, "\sim"] & \mathbf{FEt}_{X^{\mathrm{an}}} \arrow[d, "\sim"] \\ \mathbf{FEt}_{X_{\mathrm{red}}} \arrow[r, "\sim"] & \mathbf{FEt}_{X^{\mathrm{an}}_{\mathrm{red}}} \end{tikzcd}$$

   gives the required equivalence of categories.

3. We may assume X is normal.

   Any reduced scheme admits a normalisation $p : \tilde{X} \to X$ where $\tilde{X}$ is a normal scheme. By [SGA I, Exposé IX], p is a morphism of effective descent for the category $\mathbf{FEt}_X$, that is to say, the induced functor

$$\mathbf{FEt}_X \to \mathbf{FEt}_{\tilde{X}}$$

   is an equivalence of categories. Adapting the proof in [SGA I, Exposé IX] for analytic spaces, we get that p^{an} is also a morphisms of effective descent.

   Let $\tilde{\mathcal{X}}' = \mathcal{X}' \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}} \in \mathbf{FEt}_{\tilde{X}^{\mathrm{an}}}$. By hypothesis, there exists $\tilde{X}' \in \mathbf{FEt}_{\tilde{X}}$ such that $\tilde{X}'^{\mathrm{an}} \cong \tilde{\mathcal{X}}'$. Hence, there exists $X' \in \mathbf{FEt}_X$ such that $X' \times_X \tilde{X} \cong \tilde{X}'$. It then follows that

$$X'an \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}} \cong \tilde{X}'^{\mathrm{an}} \cong \tilde{\mathcal{X}}' = \mathcal{X}' \times_{X^{\mathrm{an}}} \tilde{X}^{\mathrm{an}}$$

   and the effective descent of p^{an} implies that $\mathcal{X}' \cong X'^{\mathrm{an}}$.

4. We may assume that X is regular.

   Since X is assumed to be normal, it is (R_1), hence the closed set of singular points of X has codimension $\mathrm{codim}(X, \mathrm{Sing}(X)) > 1$. Let $U = X - \mathrm{Sing}(X)$ be an open subscheme. Hence, U is a regular scheme and by Prop. 2.2(iv), U^{an} is a regular analytic space. By hypothesis, there exists $U' \in \mathbf{FEt}_U$ such that $U'^{\mathrm{an}} \cong \mathcal{X}'|_{U^{\mathrm{an}}}$.

   Claim 6.5. *If there exists a finite étale covering $X' \to X$ such that $X'|_U = U'$, then $X'^{\mathrm{an}} \cong \mathcal{X}'$.*

   Proof. By Lemma 5.5, there exist coherent algebras $\mathcal{F}$ and $\mathcal{G}$ over X^{an} corresponding to X'^{an} and $\mathcal{X}'$ respectively. Let $i^{\mathrm{an}} : U^{\mathrm{an}} \to X^{\mathrm{an}}$ be the open immersion. Since $\mathrm{codim}(X^{\mathrm{an}}, X^{\mathrm{an}} - U^{\mathrm{an}}) \geq 2$, by [Ser66, Prop. 4], the canonical maps $\mathcal{F} \to i^{\mathrm{an}}_* i^{\mathrm{an}*}\mathcal{F}$ and $\mathcal{G} \to i^{\mathrm{an}}_* i^{\mathrm{an}*}\mathcal{G}$ are isomorphisms. Hence, we get the required isomorphism from the following commutative square

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\sim"] \arrow[d] & i^{\mathrm{an}}_*\mathcal{F}|_{U^{\mathrm{an}}} \arrow[d, "\sim"] \\ \mathcal{G} \arrow[r, "\sim"] & i^{\mathrm{an}}_*\mathcal{G}|_{U^{\mathrm{an}}} \end{tikzcd}$$

   ∎

   It remains to construct such an X'. By hypothesis, X is integral affine, so let $X = \mathrm{Spec}\, A$ for some integral $\mathbb{C}$-algebra of finite type. U can be covered by a finite number of open affines $U_i = \mathrm{Spec}\, A_{f_i}$ for some $f_i \in A$. By Lemma 5.5, U' corresponds to a coherent $\mathcal{O}_U$-algebra, so on each U_i, $U'|_{U_i}$ corresponds to some A_{f_i}-algebra C_i of finite type such that $(C_i)_{f_j} \cong (C_j)_{f_i}$ for all i, j. Fixing an i and applying appropriate homomorphisms to C_j for all $j \neq i$, we can choose $C_j = A_{f_j}[x_1, \ldots, x_n]/I_j$ such that $(I_j)_{f_k} = (I_k)_{f_j} \subset K[x_1, \ldots, x_n]$ for all j, k where $K = \mathrm{Frac}\, A$.

   Let $I = A[x_1, \ldots, x_n] \cap (\cap_j I_j)$. Then, we have

$$I_{f_i} = A_{f_i}[x_1, \ldots, x_n] \cap (\cap_j (I_j)_{f_i}) = A_{f_i}[x_1, \ldots, x_n] \cap (\cap_j (I_i)_{f_j}) = I_i,$$

   so the finite cover X' of X corresponding to $A[x_1, \ldots, x_n]/I$ extends U'.

Now, we may assume that $X = \text{Spec } A$ is connected, affine and regular. We complete the proof in this case. We note that exists a compactification $j : X \to P$ of X such that P is proper and j is a dominant open immersion. P may not be regular, but we can resolve the singularities in P by taking blow ups of points in $P - X$ to obtain a proper regular scheme R over P. Results from the resolution of singularities over $\mathbb{C}$ shows that we have a dominant open immersion $k : X \to R$ such that the following diagram commutes:

$$\begin{tikzcd} & R \arrow[d, "r"] \\ X \arrow[ur, "k"] \arrow[r, "j"] & P \end{tikzcd}$$

Suppose the finite étale covering $\mathcal{X}' \to X^{\text{an}}$ can be extended to a finite (but not necessarily étale) covering $\mathcal{R}' \to R^{\text{an}}$, then by Riemann's existence theorem for proper schemes (Prop. 5.6), there exists a finite covering $R' \to R$ such that $R'^{\text{an}} \cong \mathcal{R}'$. Let $X' = R'|_X$, then

$$X'^{\text{an}} = R'|_X^{\text{an}} \cong R'^{\text{an}}|_{X^{\text{an}}} \cong \mathcal{R}'|_{X^{\text{an}}} = \mathcal{X}'.$$

It thus remains to show that $\mathcal{X}'$ can be extended to R^{an}. This problem is local on each point $x \in R^{\text{an}} - X^{\text{an}}$. Fix x. Since X^{an} and R^{an} are regular, by the implicit function theorem, there exists an open neighbourhood $V \subset R^{\text{an}}$ of x with a biholomorphic map

$$\phi : V \xrightarrow{\sim} \mathbb{C}^n \quad : \quad x \mapsto 0 \qquad \text{and} \qquad \phi(V \cap (R^{\text{an}} - X^{\text{an}})) = Z(x_1, \ldots, x_p) \subset \mathbb{C}^n$$

where $p = \text{codim}(R^{\text{an}}, R^{\text{an}} - X^{\text{an}})$. Let $U = \mathbb{C}^n$ and $U_0 = \mathbb{C}^n - Z(x_1, \ldots, x_p) = (\mathbb{C} - \{0\})^p \times \mathbb{C}^{n-p}$.

There is an equivalence between the category $\mathbf{FEt}_U$ ($\mathbf{FEt}_{U_0}$, respectively) of finite étale covers of U (U_0) and the category $\mathbf{FTopCov}_U$ ($\mathbf{FTopCov}_{U_0}$) of finite topological covers of U (U_0). The universal cover of U_0 is given by

$$\mathbb{C}^n \xrightarrow{(\exp,\ldots,\exp,\text{id},\ldots,\text{id})} (\mathbb{C} - \{0\})^p \times \mathbb{C}^{n-p}$$

and has kernel isomorphic to $\mathbb{Z}^p$. Indeed, each isomorphism class of finite topological cover of U_0 can be parametrized by the projection map

$$\begin{array}{rcl} Z(z_1 - t_1^{r_1}, \ldots, z_p - t_p^{r_p}) \subset ((\mathbb{C} - \{0\}) \times \mathbb{C})^p \times \mathbb{C}^{n-p} & \longrightarrow & (\mathbb{C} - \{0\})^p \times \mathbb{C}^{n-p} \\ (z_1, t_1, \ldots, z_p, t_p, u_1, \ldots, u_{n-p}) & \mapsto & (z_1, \ldots, z_p, u_1, \ldots, u_{n-p}) \end{array}$$

Indeed such maps also parametrize all isomorphism classes of finite étale covers of U_0. We can then see that these maps extend to a finite cover of U, ramified on $U - U_0$, given by

$$\begin{array}{rcl} Z(z_1 - t_1^{r_1}, \ldots, z_p - t_p^{r_p}) \subset (\mathbb{C} \times \mathbb{C})^p \times \mathbb{C}^{n-p} & \longrightarrow & \mathbb{C}^p \times \mathbb{C}^{n-p} \\ (z_1, t_1, \ldots, z_p, t_p, u_1, \ldots, u_{n-p}) & \mapsto & (z_1, \ldots, z_p, u_1, \ldots, u_{n-p}) \end{array}$$

This completes the proof of the theorem. ☐

Recall that the étale fundamental group $\pi(X, x)$ of a scheme X at a point $x \in X$ is defined to be the automorphism group F_x where $F_x : \mathbf{FEt}_X \to \mathbf{FSets}$ is the functor sending each finite étale covering $X' \xrightarrow{f} X$ to the fibre $f^{-1}(x)$. We can similarly define the étale fundamental group of an analytic space. Riemann's existence theorem gives us the following:

Corollary 6.6. *Let X be a $\mathbb{C}$-scheme locally of finite type and $x \in X^{\text{an}}$. Then, the étale fundamental group $\pi(X, x)$ of X at x is isomorphic to the profinite completion $\pi_1\widehat{(X^{\text{an}}, x)}$ of the topological fundamental group of X^{an} at x.*

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathbf{FEt}_X \arrow[r, "F_x"] \arrow[d, "\Phi"] & \mathbf{FSets} \arrow[d, "\text{id}"] \\ \mathbf{FEt}_{X^{\text{an}}} \arrow[r, "F_x^{\text{an}}"] & \mathbf{FSets} \end{tikzcd}$$

Since Φ is an equivalence of categories by Thm. 6.4, it induces a natural isomorphism between the functors F_x and F_x^{an}. Hence, $\pi(X,x) = \mathrm{Aut}(F_x) \cong \mathrm{Aut}(F_x^{\mathrm{an}}) = \pi(X^{\mathrm{an}}, x)$.

Since $\mathbf{FEt}_{X^{\mathrm{an}}} \cong \mathbf{FTopCov}_{X^{\mathrm{an}}}$, we get that

$$\mathrm{Aut}(\mathcal{X}' \to X^{\mathrm{an}}) \cong \pi_1(X^{\mathrm{an}}, x)/\pi_1(\mathcal{X}', x).$$

Hence,

$$\pi(X,x) \cong \mathrm{Aut}(F_x^{\mathrm{an}}) \cong \varprojlim_{(\mathcal{X}' \to X^{\mathrm{an}}) \in \mathbf{FEt}_{X^{\mathrm{an}}}} \pi_1(X^{\mathrm{an}}, x)/\pi_1(\mathcal{X}', x) = \pi_1\widehat{(X^{\mathrm{an}}}, x).$$

□

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