Cohomology on quasi-coherents, torsors, H^1 and the Picard group

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These are notes for a talk held at the étale cohomology seminar in Leiden, The Netherlands, on Tuesday 21 October 2014. The author apologises for all errors, unclarities, omissions of details and other imperfections and encourages the reader to send them by email to the author: r.van.bommel@math.leidenuniv.nl. The notes are mainly based on [Stacks] and [Edix11]. For more details the reader can consult [Giraud], for example.

1 Quasi-coherent sheaves

Definition 1 (Ringed site, [Stacks, 04KQ])**.** A *ringed site* is a pair $(\mathcal{C}, \mathcal{O})$ of a site $\mathcal{C}$ and a sheaf of rings $\mathcal{O}$ on $\mathcal{C}$, also called the *structure sheaf* of the ringed site.

Example 2. Let $\mathcal{C}$ be the (small or large) (Zariski or étale) site over any scheme S. The presheaf

$$\mathcal{C} \to \mathrm{Ab} : T \mapsto \mathcal{O}_T(T)$$

is a sheaf of rings on $\mathcal{C}$ (cf. prop. 2.5 of Zomervrucht's talk). We choose this sheaf to be the structure sheaf and we denote it by $\mathcal{O}$, $\mathcal{O}_S$ or $\mathcal{O}_{S,\mathcal{C}}$ depending on the context. Then $(\mathcal{C}, \mathcal{O})$ is a ringed site.

Example 3. If $(\mathcal{C}, \mathcal{O})$ is a ringed site and $U \in \mathcal{C}$ an object, then we can consider the site $\mathcal{C}/U$ consisting of objects of $\mathcal{C}$ with a morphism to U (with the same coverings as in $\mathcal{C}$). The sheaf $\mathcal{O}$ can be restricted to a sheaf $\mathcal{O}_U$ on $\mathcal{C}/U$ and $(\mathcal{C}/U, \mathcal{O}_U)$ is a ringed site, called *the localisation of* $(\mathcal{C}, \mathcal{O})$ *at* U.

Definition 4 (Quasi-coherent sheaf on a site, [Stacks, 03DL])**.** A sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on a ringed site $(\mathcal{C}, \mathcal{O})$ is called *quasi-coherent* if for every $U \in \mathcal{C}$ there is a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{\mathcal{C}/U_i}$ is an $\mathcal{O}_{U_i}$-module for which there exists an exact sequence

$$\bigoplus_J \mathcal{O}_{U_i} \longrightarrow \bigoplus_K \mathcal{O}_{U_i} \longrightarrow \mathcal{F}|_{\mathcal{C}/U_i} \longrightarrow 0$$

of $\mathcal{O}_{U_i}$-modules.

Theorem 5. *Generalising the construction in proposition 2.5 of Zomervrucht's talk we define for any quasi-coherent sheaf $\mathcal{F}$ on S the sheaf $\mathcal{F}_{\mathcal{C}} = (T \mapsto \mathcal{F}_T(T))$ on the ringed site $\mathcal{C} = S_{ét}, S_{\mathrm{Zar}}, (Sch/S)_{ét}, (Sch/S)_{\mathrm{Zar}}$. This sheaf is quasi-coherent. Furthermore, the cohomology groups are the same:*

$$H^p(S, \mathcal{F}) = H^p(\mathcal{C}, \mathcal{F}_{\mathcal{C}}).$$

Proof. This is difficult, for a proof see for example [Stacks, 03DX] and [Stacks, 03P2]. □

2 Torsors and H^1

This section is based on notes from Edixhoven's talk in the étale cohomology seminar three years ago, see [Edix11], and on sections [Stacks, 03AG] and [Stacks, 040D] of the Stacks project. In this section we will prove that for any abelian sheaf $\mathcal{G}$, the set $H^1(\mathcal{C}, \mathcal{G})$ is canonically isomorphic to the set of isomorphism classes of $\mathcal{G}$-torsors.

Definition 6 (Torsor, [Stacks, 03AH])**.** Let $\mathcal{C}$ be a site and $\mathcal{G}$ be a sheaf of groups on $\mathcal{C}$. Then a *$\mathcal{G}$-torsor* is a sheaf of sets $\mathcal{F}$ on $\mathcal{C}$ with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$, such that for all $U \in \mathcal{C}$ the following holds: the action of $\mathcal{G}(U)$ on $\mathcal{F}(U)$ is free (i.e. all point stabilizers are trivial) and transitive and there exists a cover $\{U_i \to U\}$ of U such that $\forall i : \mathcal{F}(U_i) \neq \varnothing$. Another way to phrase this last condition: $\mathcal{F}$ is locally isomorphic to $\mathcal{G}$.

A *morphism of $\mathcal{G}$-torsors* is a morphism of sheaves of sets that respects the action of $\mathcal{G}$. A $\mathcal{G}$-torsor is called *trivial* if it is isomorphic to $\mathcal{G}$ as $\mathcal{G}$-torsor.

Remark 7. The category of $\mathcal{G}$-torsors is a groupoid, i.e., all morphisms are isomorphisms (proof: check this locally). A $\mathcal{G}$-torsor $\mathcal{F}$ is trivial if and only if

$$\varnothing \neq \mathcal{F}(\mathcal{C}) := \lim_{\mathcal{C}} \mathcal{F} = \left\{ (s_X)_X \in \prod_{X \in \mathcal{C}} \mathcal{F}(X) \text{ compatible} \right\},$$

in that case $\mathcal{G} \to \mathcal{F} : g \mapsto g \cdot x$ is an isomorphism for any $x \in \mathcal{F}(\mathcal{C})$.

Example 8 ([Edix11])**.** Let S be a scheme for which $n \in \mathbb{Z}_{\geqslant 1}$ is in $\mathcal{O}(S)^*$. Consider $\mu_{n,S} = \ker(\cdot^n : \mathbb{G}_{m,S} \to \mathbb{G}_{m,S})$ in the big étale site $(\mathrm{Sch}/S)_{\text{ét}}$. Then for every $a \in \mathcal{O}_S(S)^*$, the fibre $F_a = (\cdot^n)^{-1}\{a\}$ is a $\mu_{n,S}$-torsor.

Let us illustrate this in the case $S = \operatorname{Spec} A$ is affine. Then for any A-algebra $f : A \to B$ we have that the fibre is $F_a(B) = \{b \in B^* : b^n = f(a)\}$. The group $\mu_{n,A}(B) = \{\zeta \in B^* : \zeta^n = 1\}$ acts on it by multiplication. This action is transitive as $\frac{b}{b'} \in \mu_{n,A}(B)$ for all $b, b' \in F_a(b)$. It is free as every $b \in F_a(B)$ is a unit. Remark that $B \to C := B[x]/(x^n - f(a))$ is standard étale as $n \cdot \overline{x}^{n-1} \in C$ is invertible. As $F_a(C) \neq \varnothing$ this proves that $\mu_{n,A}$ is non-empty on the étale cover C of B.

Definition 9. Let $\mathcal{C}$ be a site and $\mathcal{G}$ be a sheaf of groups on $\mathcal{C}$. Suppose that we have sheaves $\mathcal{X}$ and $\mathcal{Y}$ on $\mathcal{C}$ with a right- and left action of $\mathcal{G}$, respectively. Then we define the sheaf

$$\mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y} = (T \mapsto (\mathcal{X}(T) \times \mathcal{Y}(T))/\mathcal{G}(T))^{\#},$$

where $\mathcal{G}$ acts on the right on $\mathcal{X} \times \mathcal{Y}$ by $(x,y)g = (xg, g^{-1}y)$. You can also give a universal property for this sheaf $\mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y}$: any morphism of sheaves $\psi : \mathcal{X} \times \mathcal{Y}$ that is $\mathcal{G}$-bilinear, i.e., $\forall g, x, y : \psi(xg, y) = \psi(x, gy)$, factors uniquely through the map $\mathcal{X} \times \mathcal{Y} \to \mathcal{X} \otimes_{\mathcal{G}} \mathcal{Y}$. This object is also called the *contracted producted* and many people use the notation $\mathcal{X} \wedge^{\mathcal{G}} \mathcal{Y}$ to denote it.

Remark 10. Let $\mathcal{C}$ be a site. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are locally isomorphic sheaves of sets on $\mathcal{C}$ (or locally free $\mathcal{O}$-modules of rank n or your other favourite type of objects with the appropriate properties). Let $\mathcal{I} = \mathscr{I}som(\mathcal{X}, \mathcal{Y})$ and $\mathcal{G} = \mathscr{A}ut(\mathcal{X})$ and let $\mathcal{G}$ act on $\mathcal{I}$ on the right and on $\mathcal{X}$ on the left in the natural way. Then the map

$$\mathcal{I} \times \mathcal{X} \to \mathcal{Y} : (i, x) \mapsto i(x)$$

is $\mathcal{G}$-bilinear and it induces a morphism $\mathcal{I} \otimes_{\mathcal{G}} \mathcal{X} \to \mathcal{Y}$. We can construct an inverse locally by $y \mapsto i \otimes (i^{-1}y)$ for any $i \in \mathcal{I}$, we can show that this does not depend on the choice of i and, hence, that it glues to an inverse morphism $\mathcal{Y} \to \mathcal{I} \otimes_{\mathcal{G}} \mathcal{X}$. Instead of constructing the inverse, we could also use the universal property to prove that $\mathcal{I} \otimes_{\mathcal{G}} \mathcal{X} \cong \mathcal{Y}$, see exercise 11. The morale is that $\mathscr{A}ut(\mathcal{X})$ naturally acts on $\mathcal{X}$ and to get $\mathcal{Y}$ from it you need to twist it by the $\mathscr{A}ut(\mathcal{X})$-torsor $\mathscr{I}som(\mathcal{X}, \mathcal{Y})$. In the proof of theorem 12 we will use this on the Ext^1.

Exercise 11. Check that the $\mathcal{G}$-bilinear map $\mathcal{I} \times \mathcal{X} \to \mathcal{Y}$ defined in remark 10 satisfies the universal property.

Theorem 12 ([Stacks, 03AJ] & [Edix11])**.** *Let $\mathcal{C}$ be a ringed site and let $\mathcal{G}$ be an $\mathcal{O}$-module on $\mathcal{C}$. Then there is a canonical bijection*

$$\{\mathcal{G}\text{-}torsors\}/\cong \;\leftrightarrow H^1(\mathcal{C}, \mathcal{G}).$$

Remark 13. The final result of theorem 12 does not depend on the choice of $\mathcal{O}$! Hence, if $\mathcal{G}$ is an arbitrary sheaf of abelian groups on $\mathcal{C}$, we could choose $\mathcal{O}$ to be the constant sheaf of rings $\underline{\mathbb{Z}}$ and the theorem can be applied.

In fact, it is possible to define a group structure on the set of isomorphism classes of $\mathcal{G}$-torsors and then this bijection will be an isomorphism of groups, see for example [Edix11].

Proof. This proof is based on the proof of [Edix11]. The author would like to thank him for sharing and explaining this proof. In this proof the H^1 is first proved to be in bijection with the Ext^1 and then the latter is proved to be in bijection with the set of isomorphism classes of $\mathcal{G}$-torsors.

For every $\mathcal{O}$-module $\mathcal{F}$ there is a canonical natural isomorphism

$$H^0(\mathcal{C}, \mathcal{F}) = \lim_{\mathcal{C}} \mathcal{F} = \left\{ (s(X))_X \in \prod_{X \in \mathcal{C}} \mathcal{F}(X) \text{ compatible} \right\} = \dots$$

$$\dots = \left\{ (f_X)_X \in \prod_{X \in \mathcal{C}} \mathrm{Hom}(\mathcal{O}(X), \mathcal{F}(X)) \text{ compatible} \right\} = \mathrm{Hom}_{\mathcal{O}-\mathrm{mod}}(\mathcal{O}, \mathcal{F}).$$

Hence, $H^1(\mathcal{C}, \mathcal{G}) = (R^1\mathrm{Hom}(\mathcal{O}, -))(\mathcal{G}) =: \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}, \mathcal{G})$.

The next lemma we will use in the case of the category of $\mathcal{O}$-modules on $\mathcal{C}$.

Lemma 14. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then we have* $\mathrm{Ext}^1(A,B) = \{B \hookrightarrow E \twoheadrightarrow A\}/\cong$, *where two extensions* $B \hookrightarrow E \twoheadrightarrow A$ *and* $B \hookrightarrow E' \twoheadrightarrow A$ *are isomorphic if there exists a diagram as follows.*

$$\begin{array}{ccccc} B & \hookrightarrow & E & \twoheadrightarrow & A \\ \| & & \downarrow{\scriptstyle f} & & \| \\ B & \hookrightarrow & E' & \twoheadrightarrow & A \end{array}$$

Remark 15. In such a diagram the map f automatically is an isomorphism, because of the 5-lemma.

Proof. Let $B \hookrightarrow I$ be an injective map to an injective object and let Q be the quotient. Then consider the long exact sequence:

$$\mathrm{Hom}(A,B) \to \mathrm{Hom}(A,I) \to \mathrm{Hom}(A,Q) \to \mathrm{Ext}^1(A,B) \to 0 = \mathrm{Ext}^1(A,I).$$

We will construct a bijection $\mathrm{Hom}(A,Q)/\mathrm{Im}(\mathrm{Hom}(A,I)) \to \{B \hookrightarrow E \twoheadrightarrow A\}/\cong$.

Exercise 16. Prove that finite fibered (co)products exist in any abelian category and describe them explicitly.

Given a homomorphism $f : A \to Q$, consider the base change $E_f = A \times_Q I$. We get an extension in the following way.

$$\begin{array}{ccccc} B & \hookrightarrow & I & \twoheadrightarrow & Q \\ \| & & \uparrow{\scriptstyle p_f} & & \uparrow{\scriptstyle f} \\ B & \hookrightarrow & E_f & \overset{q_f}{\twoheadrightarrow} & A \end{array}$$

If you have two extensions $B \hookrightarrow E \twoheadrightarrow A$ and $B \hookrightarrow E' \twoheadrightarrow A$, we can define their sum in the following way. By taking the direct sum we get an exact sequence $B \oplus B \hookrightarrow E \oplus E' \twoheadrightarrow A \oplus A$.

Now consider the diagonal map $A \to A \oplus A$ and let E be the fibered product $(E_1 \oplus E_2) \times_{A \oplus A} A$. This gives us an exact sequence $B \oplus B \hookrightarrow E \twoheadrightarrow A$.

Now let E' be the push-out of the maps $B \oplus B \hookrightarrow E$ and the addition map $B \oplus B \to B$. This gives us an exact sequences $B \hookrightarrow E' \twoheadrightarrow A$, which we define to be the sum of $B \hookrightarrow E \twoheadrightarrow A$ and $B \hookrightarrow E' \twoheadrightarrow A$.

The following diagram summarises the construction.

$$\begin{array}{ccccc} B \oplus B & \hookrightarrow & E_1 \oplus E_2 & \twoheadrightarrow & A \oplus A \\ \| & & \uparrow & & \uparrow{\scriptstyle \Delta} \\ B \oplus B & \hookrightarrow & E & \twoheadrightarrow & A \\ \downarrow{\scriptstyle +} & & \downarrow & & \| \\ B & \hookrightarrow & E' & \twoheadrightarrow & A \end{array}$$

Exercise 17. Show that this sum construction gives $\{B \hookrightarrow E \twoheadrightarrow A\}/\cong$ the structure of an abelian group and that the map $\mathrm{Hom}(A,Q) \to \{B \hookrightarrow E \twoheadrightarrow A\}/\cong$ constructed in the first paragraph of this proof is a morphism of groups.

Now let us calculate the kernel of this morphism $\mathrm{Hom}(A,Q) \to \{B \hookrightarrow E \twoheadrightarrow A\}/\cong$. The identity element of $\{B \hookrightarrow E \twoheadrightarrow A\}/\cong$ is the split sequence. So, for which $f \in \mathrm{Hom}(A,Q)$ does E_f split? The answer is not hard, it splits if and only if we have a section $A \to E_f$ of q_f, which we have if and only if the map $f : A \to Q$ factors via $I \to Q$. Hence, we get an injective homomorphism $\mathrm{Hom}(A,Q)/\mathrm{Im}(\mathrm{Hom}(A,I)) \to \{B \hookrightarrow E \twoheadrightarrow A\}/\cong$.

Now we will prove that this morphism is surjective. Let $B \hookrightarrow E \twoheadrightarrow A$ be an extension. Then we can extend it to a diagram

$$\begin{array}{ccccc} B & \hookrightarrow & E & \twoheadrightarrow & A \\ \| & & \downarrow & & \downarrow{\scriptstyle f} \\ B & \hookrightarrow & I & \twoheadrightarrow & Q \end{array}$$

to get a morphism $f : A \to Q$. Then by formal nonsense we get a morphism $E \to E_f$ (of extensions), which automatically is an isomorphism. □

Now we finish the proof of theorem 12. We will prove that $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O},\mathcal{G})$ is canonically isomorphic with the set of isomorphism classes of $\mathcal{G}$-torsors.

Suppose that $\mathcal{G} \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{O}$ is an extension of $\mathcal{O}$-modules. Locally the map splits, as $\mathcal{O}$ is a free $\mathcal{O}$-module, i.e., it is locally isomorphic to $\mathcal{G} \hookrightarrow \mathcal{G} \oplus \mathcal{O} \twoheadrightarrow \mathcal{O}$. The sheaf $\mathscr{I}\!som_{\mathrm{Ext}}(\mathcal{G} \oplus \mathcal{O}, \mathcal{E})$ can be supplied with the structure of a $\mathcal{G}$-torsor by remarking that the map

$$\mathcal{G} \to \mathscr{A}\!ut_{\mathrm{Ext}}(\mathcal{G} \oplus \mathcal{O}) : g \mapsto ((g',x) \mapsto (g' + x \cdot g, x))$$

is an isomorphism and use the natural action of the latter sheaf. Or, to put it in an informal way, $g \in \mathcal{G}$ acts on $\mathcal{G} \oplus \mathcal{O}$ by means of the following matrix:

$$\begin{pmatrix} \mathrm{Id} & g \\ 0 & \mathrm{Id} \end{pmatrix}.$$

Let us also illustrate this with a diagram.

$$\begin{array}{ccccc} \mathcal{G} & \longrightarrow & \mathcal{G} \oplus \mathcal{O} & \longrightarrow & \mathcal{O} \\ \| & & \downarrow{\scriptstyle \chi} & & \| \\ \mathcal{G} & \longrightarrow & \mathcal{G} \oplus \mathcal{O} & \longrightarrow & \mathcal{O} \end{array}$$

Now χ must map $(g',0)$ to $(g',0)$ for all $g' \in \mathcal{G}(X)$ and $(0,1)$ to $(g,1)$ for some $g \in \mathcal{G}(X)$. The rest of the map is given by the $\mathcal{O}$-module structure.

To summarize, we can associate a $\mathcal{G}$-torsor to an extension. On the other hand, if we have a $\mathcal{G}$-torsor $\mathcal{T}$ we can associate to it an extension

$$\mathcal{G} \hookrightarrow \mathcal{T} \otimes_{\mathcal{G}} (\mathcal{G} \oplus \mathcal{O}) \twoheadrightarrow \mathcal{O}$$

in the following way. We apply the functor $\mathcal{T} \otimes_{\mathcal{G}} (-)$ to the canonical sequence $\mathcal{G} \hookrightarrow \mathcal{G} \oplus \mathcal{O} \twoheadrightarrow \mathcal{O}$ of sheaves with an action $\mathcal{G}$, where $\mathcal{G}$ acts trivially on $\mathcal{G}$ (really trivial in the sense of the trivial action, not in the sense of a trivial $\mathcal{G}$-torsor) and $\mathcal{O}$, and it acts on $\mathcal{G} \oplus \mathcal{O}$ as before (in particular not trivially):

$$\mathcal{T} \otimes_{\mathcal{G}} \mathcal{G} \hookrightarrow \mathcal{T} \otimes_{\mathcal{G}} (\mathcal{G} \oplus \mathcal{O}) \twoheadrightarrow \mathcal{T} \otimes_{\mathcal{G}} \mathcal{O}.$$

Remark that the projection $\mathcal{T} \times \mathcal{O} \to \mathcal{O}$ is $\mathcal{G}$-equivariant as $\mathcal{G}$ acts trivially on $\mathcal{O}$. Hence, we get a morphism $\mathcal{T} \otimes_{\mathcal{G}} \mathcal{O} \to \mathcal{O}$. The sheaf $\mathcal{T}$ is, as a $\mathcal{G}$-torsor, locally isomorphic to $\mathcal{G}$ (with the multiplicative action) and hence this morphism is an isomorphism. In the same way $\mathcal{T} \otimes_{\mathcal{G}} \mathcal{G} = \mathcal{G}$ as $\mathcal{G}$ also acts trivially on $\mathcal{G}$ in our exact sequence. This finishes our construction.

To prove that these constructions are quasi-inverse it sufficed to prove that we have a morphism

$$\mathcal{T} \to \mathscr{I}\!som_{\mathrm{Ext}}(\mathcal{G} \oplus \mathcal{O}, \mathcal{T} \otimes_{\mathcal{G}} (\mathcal{G} \oplus \mathcal{O}))$$

$$t \mapsto ((g, x) \mapsto t \otimes (g, x))$$

of $\mathcal{G}$-torsors which then automatically is an isomorphism. Furthermore, we can use remark 10 to prove that the other composition is naturally isomorphic to the identity, i.e., to get an isomorphism

$$\mathcal{E} \cong \mathscr{I}\!som_{\mathrm{Ext}}(\mathcal{G} \oplus \mathcal{O}, \mathcal{E}) \otimes_{\mathcal{G}} (\mathcal{G} \oplus \mathcal{O}).$$

This concludes the proof of the equivalence and the theorem. □

3 Picard group

Definition 18 (Picard group, [Stacks, 040C])**.** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Then the *Picard group* $\mathrm{Pic}(\mathcal{O})$ of the ringed site is the abelian group consisting of isomorphism classes of $\mathcal{O}$-modules locally isomorphic to $\mathcal{O}$ (locally free of rank 1, invertible) with the tensor product as group operation.

Theorem 19 ([Stacks, 040E])**.** *Let $(C, \mathcal{O})$ be a ringed site. There is a canonical isomorphism*

$$H^1(\mathcal{C}, \mathcal{O}^*) = \mathrm{Pic}(\mathcal{O})$$

of abelian groups.

Proof. After everything we did in the previous section this proof is fairly simple. Identify $H^1(\mathcal{C}, \mathcal{O}^*)$ with the set of isomorphism classes of $\mathcal{O}^*$-torsors on $\mathcal{C}$ using theorem 12. Then consider the following constructions.

$$\{\mathcal{O}^*\text{-torsors}\}/\cong \;\leftrightarrow\; \mathrm{Pic}(\mathcal{O})$$

$$T \mapsto T \otimes_{\mathcal{O}^*} \mathcal{O}$$

$$\mathscr{I}\!som_{\mathcal{O}}(\mathcal{O}, \mathcal{L}) \leftarrow \mathcal{L}$$

Here we remark that the action of $\mathcal{O}^*$ on $\mathcal{O}$ is given by the identification $\mathcal{O}^* = \mathscr{A}ut_{\mathcal{O}}(\mathcal{O})$. To check that these constructions are mutually inverse we need to check that

$$T \cong \mathscr{I}som_{\mathcal{O}}(\mathcal{O}, T \otimes_{\mathcal{O}^*} \mathcal{O})$$
$$\mathcal{L} \cong \mathscr{I}som_{\mathcal{O}}(\mathcal{O}, \mathcal{L}) \otimes_{\mathcal{O}^*} \mathcal{O}.$$

The former follows from the fact that $t \mapsto (x \mapsto (t \otimes x))$ is an isomorphism. The latter follows again from remark 10, just like in the proof of theorem 12. □

Exercise 20. Let $(\mathcal{C}, \mathcal{O})$ be a ringed site. Prove that there is a canonical bijection between the set of isomorphism classes of locally free $\mathcal{O}$-modules of rank n and the set of isomorphism classes of $\mathrm{GL}_n(\mathcal{O})$-torsors.

Theorem 21 ([Stacks, 03P7])**.** *Let X be a scheme. Then there are canonical identifications*

$$\begin{aligned} H^1((\mathrm{Sch}/X)_{ét}, \mathbb{G}_m) = H^1((\mathrm{Sch}/X)_{\mathrm{Zar}}, \mathbb{G}_m) = H^1(X_{ét}, \mathbb{G}_m) = H^1(X_{\mathrm{Zar}}, \mathbb{G}_m) \\ = H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X). \end{aligned}$$

Proof. As seen in theorem 19 there is a canonical identification between the H^1 and the Picard group on the site. To prove that the Picard groups are isomorphic, note that every quasi-coherent $\mathcal{O}_{X,\text{ét}}$-module (resp. $\mathcal{O}_{\mathrm{Zar}}$) descends to a quasicoherent $\mathcal{O}_X$-module and so do the morphisms between them (it is an equivalence of categories, see for example [Stacks, 03DX]). Hence, locally free rank 1 modules descend to locally free rank 1 modules and the notion of isomorphism descends in a faithful way. In other words, we find identifications $\mathrm{Pic}(\mathcal{O}_{X,\text{ét}}) = \mathrm{Pic}(\mathcal{O}_{X,\mathrm{Zar}}) = \mathrm{Pic}(X)$. □

References

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