Étale cohomology over points and curves

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1 Étale cohomology over a point

Definition 1.1. Let G be a topological group.
The category G-Sets has:

- as objects, sets X endowed with a left action $G \times X \to X$ of the group G, so that the action is continuous when X is given the discrete topology and $G \times X$ the product topology;
- G-equivariant maps as morphisms.

Remark 1.2. The condition on continuity of the action amounts to requesting that for every $x \in X$, the stabilizer of x is open in G.

If $K \hookrightarrow L$ is a Galois extension of fields, the Galois group $\mathrm{Gal}(L|K)$ is a topological group, whose open subgroups are of the form $\mathrm{Gal}(L|K')$ with $K \hookrightarrow K' \hookrightarrow L$ a subextension of K with $[K' : K] < \infty$.

Lemma 1.3 (Stacks,TAG 03QR)**.** *Let K be a field and $K \hookrightarrow K^{sep}$ an inclusion in a separable closure of K. Let $G := \mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}|K)$ be the absolute Galois group of K.*
The functor

$$\begin{array}{rcl} \textit{Schemes étale over } K & \to & G\textit{-Sets} \\ X/K & \mapsto & \mathrm{Hom}_K(\mathrm{Spec}\,\mathrm{K}^{\mathrm{sep}}, X) \end{array}$$

is an equivalence of categories.

Sketch of proof. A scheme X is étale over K if and only if $X \cong \bigsqcup_{i \in I} \mathrm{Spec}\, K_i$ with each K_i a finite separable extension of K. The functor in the statement associates to X the set

$$\mathrm{Hom}_K(\mathrm{Spec}\,\mathrm{K}^{\mathrm{sep}}, \sqcup_{i \in I} \mathrm{Spec}\, K_i) = \bigsqcup_i \mathrm{Hom}_K(K_i, \mathrm{K}^{\mathrm{sep}})$$

which is endowed with a natural left G-action. Moreover, any element of this set is stabilized by an open subgroup, so we really obtain a G-set. To find a quasi-inverse to this functor, observe that any G-set S is a disjoint union of its orbits $(S_j)_{j \in J}$. Pick an element $s_j \in S_j$ from each orbit. Let $H_j \subset G$ be the open stabilizer of s_j. Then the quasi-inverse functor is given by

$$S \mapsto \bigsqcup_{j \in J} \mathrm{Spec}(\mathrm{K}^{\mathrm{sep}})^{H_j}.$$

I omit to check that the functors are indeed quasi-inverse. One remark: different choices of s_j's lead to isomorphic schemes. Indeed, one gets new subgroups H_j that are conjugated to the previous ones. Now notice that $(K^{sep})^{H_j} \cong g((K^{sep})^{H_j}) = (K^{sep})^{gHg^{-1}}$ for any $g \in G$.

□

In the first lecture of this seminar, Jinbi defined, for any sheaf $\mathcal{F}$ on the small étale site over a scheme S, and for any geometric point $\overline{s}$ of S, the stalk $\mathcal{F}_{\overline{s}}$ as

$$\mathcal{F}_{\overline{s}} := \operatorname{colim}_{(U,\overline{u})} \mathcal{F}(U)$$

where the colimit runs over the étale neighbourhoods of $\overline{s}$. We show that if $\overline{s}$ lies over $s \in S$, there is an action of the absolute Galois group of $k(s)$ on the stalk $\mathcal{F}_{\overline{s}}$. Indeed, elements of $\mathcal{F}_{\overline{s}}$ are 3-tuples $(U, \overline{u}, t)$ up to equivalence, with $(U, \overline{u})$ an étale neighbourhood of $\overline{s}$ and $t \in \mathcal{F}(U)$. Any $\sigma \in \operatorname{Aut}_{k(s)}(k(\overline{s}))$ acts on $\mathcal{F}_{\overline{s}}$ by

$$(U, \overline{u}, t) \mapsto (U, \overline{u} \circ \operatorname{Spec} \sigma, t).$$

If σ is the identity when restricted to the separable closure $k(s)^{sep}$ of $k(s)$ inside $k(\overline{s})$, then σ acts trivially on $\mathcal{F}_{\overline{s}}$. Therefore the action of $\operatorname{Aut}_{k(s)}(k(\overline{s}))$ factors via an action of $G := \operatorname{Gal}(k(s)^{sep}|k(s))$. Also, any element $(U, \overline{u}, t)$ is stabilized by the open subgroup of automorphisms of $k(s)^{sep}$ fixing the image of $k(u)$ in $k(s)^{sep}$, where u is the point of U over which $\overline{u}$ lies. Hence $\mathcal{F}_{\overline{s}}$ is naturally a G-set.
The operation of taking the stalk at a geometric point $\overline{s}$ of S thus constitutes a functor from the category $\operatorname{Sh}(S_{\text{ét}})$ of set-valued sheaves to the category of G-sets. It turns out to be an equivalence of categories!

Theorem 1.4 (Stacks,TAG 03QT). *Let K be a field and $S = \operatorname{Spec} K$. Let $\overline{s}$ be a geometric point of S and $G = \operatorname{Gal}(K^{sep}|K)$. Then the functor*

$$\begin{array}{rcl} Sh(S_{\text{ét}}) & \to & G\text{-}Sets \\ \mathcal{F} & \mapsto & \mathcal{F}_{\overline{s}} \end{array}$$

is an equivalence of categories.

Notice that if we precompose this functor with the Yoneda embedding from the category of étale schemes over S to $\operatorname{Sh}(S_{\text{ét}})$, we obtain the equivalence of categories of Lemma 1.3. Indeed an étale scheme U/S is sent first to the sheaf h_U and then to $h_{U,\overline{s}} = \operatorname{Hom}_K(\operatorname{Spec} K^{sep}, U)$. Therefore, proving the theorem is equivalent to showing that the Yoneda embedding is an equivalence of categories in this particular case. This is the result of the following proposition.

Proposition 1.5. *Let K be a field and $S = \operatorname{Spec} K$. Then every sheaf of sets $\mathcal{F}$ on the small étale site $S_{\text{ét}}$ is representable by a scheme étale over S. That is, the fully faithful functor*

$$\begin{array}{rcl} \text{Schemes étale over } S & \to & \operatorname{Sh}(S_{\text{ét}}) \\ X/S & \mapsto & h_X : (U \mapsto \operatorname{Hom}(U, X)) \end{array}$$

is actually an equivalence of categories.

Proof. We build a quasi-inverse to the Yoneda functor. Let first $\overline{s}$ be a geometric point of S and $G = \operatorname{Gal}(K^{sep}|K)$ with $K \subset K^{sep} \subset k(\overline{s})$. Given a sheaf $\mathcal{F}$, the stalk $\mathcal{F}_{\overline{s}}$ is a G-set. Picking an element s_i from each orbit of $\mathcal{F}_{\overline{s}}$, we obtain subgroups H_i of finite index in G as stabilizers of the s_i's. Then we claim that the functor $\mathcal{F} \mapsto X_{\mathcal{F}} := \sqcup_i \operatorname{Spec}(K^{sep})^{H_i}$ provides the quasi-inverse functor. We just need to show that $h_{X_{\mathcal{F}}} \cong \mathcal{F}$. In fact we only need to give a bijection $h_{X_{\mathcal{F}}}(U) \cong \mathcal{F}(U)$ for $U \cong \operatorname{Spec} L$ with L a finite separable extension of K.

The main ingredient in the proof is that, for any $K \subset L$ finite separable extension and $L \subset M$ finite Galois extension inside K^{sep}, we have

$$\mathcal{F}(\operatorname{Spec} L) = \mathcal{F}(\operatorname{Spec} M)^{\operatorname{Gal}(K^{sep}|L)}$$

For this we use the sheaf property. Seeing $\mathcal{F}$ as a covariant functor from the category of finite separable extensions of K, we have the usual equalizer sequence

$$0 \to \mathcal{F}(L) \to \mathcal{F}(M) \rightrightarrows \mathcal{F}(M \otimes_L M).$$

Letting $H = \operatorname{Gal}(M|L)$, we have $L = M^H$. Hence $M \otimes_L M \cong \prod_{h \in H} M$, the isomorphism being given by $m_1 \otimes m_2 \mapsto (m_1 h_i m_2)_{i=1}^n$ and one can in fact rewrite the sequence above as

$$0 \to \mathcal{F}(L) \to \mathcal{F}(M) \rightrightarrows \prod_{h \in H} \mathcal{F}(M)$$

where the two parallel arrows are $(\mathrm{id}, \mathrm{id}, \dots, \mathrm{id})$ and $x \mapsto (h_1 x, h_2 x, \dots, h_n x)$ respectively, for $H = \{h_1, \dots, h_n\}$. Hence it follows that $\mathcal{F}(L) = \mathcal{F}(M)^H = \mathcal{F}(M)^{\operatorname{Gal}(K^{sep}|L)}$ which was our claim.
Now it's clear that

$$\mathcal{F}(\operatorname{Spec} L) = \operatorname*{colim}_{L \subset M \text{ finite Galois}} \mathcal{F}(M)^{\operatorname{Gal}(K^{sep}|L)} = \mathcal{F}_{\overline{s}}^{\operatorname{Gal}(K^{sep}|L)}.$$

By definition of $X_{\mathcal{F}}$, we also have $\mathcal{F}_{\overline{s}} = \operatorname{Hom}(\operatorname{Spec} K^{sep}, X_{\mathcal{F}})$. Therefore

$$\mathcal{F}(\operatorname{Spec} L) = \operatorname{Hom}_K(\operatorname{Spec} K^{sep}, X_{\mathcal{F}})^{\operatorname{Gal}(K^{sep}|L)} = \operatorname{Hom}_K(\operatorname{Spec} L, X_{\mathcal{F}})$$

which is our thesis. □

This gives a proof of Theorem 1.4.

Lemma 1.6 (Stacks,TAG 04JM). *Hypoteses as in the Theorem 1.4. There is a functorial bijection*

$$\Gamma(S, \mathcal{F}) = \mathcal{F}_{\overline{s}}^G.$$

Proof.

$$\Gamma(S, \mathcal{F}) = \operatorname{Hom}_{Sh(S_{\text{ét}})}(h_S, \mathcal{F}) = \operatorname{Hom}_{G\text{-Sets}}(\{*\}, \mathcal{F}_{\overline{s}}) = \mathcal{F}_{\overline{s}}^G$$

where the first equality follows by Yoneda's lemma.
It is also proved in the previous proposition. □

Definition 1.7. Let G be a topological group. A G-module M is an abelian group endowed with a structure of G-set, so that the action $G \times M \to M$ respects the group structure of M. The category Mod_G is the category of G-modules, with morphisms given by G-equivariant group homomorphisms.

Notice that by Theorem 1.4 we immediately deduce an equivalence of categories

$$\begin{array}{ccc} \operatorname{Ab}(S_{\text{ét}}) & \to & \operatorname{Mod}_G \\ \mathcal{F} & \mapsto & \mathcal{F}_{\overline{s}} \end{array}$$

due to the fact that a sheaf of abelian groups is a sheaf of sets endowed with a commutative group law, and a sheaf of G-modules is a sheaf of G-sets also endowed with a commutative group law.

Let G be a topological group, and M a G-module. The functor

$$\begin{array}{rcl}\Gamma_G : \mathrm{Mod}_G & \to & \mathrm{Ab} \\ M & \mapsto & M^G = \{x \in M \mid gx = x \ \forall g \in G\}\end{array}$$

is left exact. Moreover, Mod_G has enough injectives, therefore Γ_G has right derived functors and we can define

$$H^i(G, M) = R^i\Gamma_G(M).$$

From exactness of the stalk functor, we deduce immediately the following.

Corollary 1.8 (Stacks,TAG 03QU)**.** *Let K be a field, $S = \operatorname{Spec} K$, $\overline{s}$ a geometric point of S and $K \subset \mathrm{K}^{\mathrm{sep}}$ the corresponding separable closure. Let $G = \mathrm{Gal}(\mathrm{K}^{\mathrm{sep}} | K)$ be the absolute Galois group of K. Let also $\mathcal{F}$ be a sheaf of abelian groups on $S_{\text{ét}}$. Then for all $q \geq 0$*

$$H^q(S_{\text{ét}}, \mathcal{F}) = H^q(G, \mathcal{F}_{\overline{s}}).$$

Proof. It follows from exactness of the stalk functor and Lemma 1.6. ☐

Example 1.9. With hypotheses as in the Corollary, we see that

i) the stalk of the constant sheaf $\underline{\mathbb{Z}/n\mathbb{Z}}_S$ is $\mathbb{Z}/n\mathbb{Z}$ with the trivial G-action;

ii) the stalk of the sheaf $\mathbb{G}_{m,S}$ is $(\mathrm{K}^{\mathrm{sep}})^\times$ with its natural G-action;

iii) the stalk of the sheaf $\mathbb{G}_{a,S}$ is $((\mathrm{K}^{\mathrm{sep}}), +)$ with its natural G-action;

iv) the stalk of the sheaf $\mu_{n,S}$ is $\mu_n(\mathrm{K}^{\mathrm{sep}})$ with its natural G-action.

As Raymond showed to us in the previous lecture of this seminar, for any scheme S we have an identification

$$\mathrm{Pic}(S) = H^1(S_{\text{ét}}, \mathbb{G}_m).$$

Also, for any scheme S and any quasi-coherent sheaf $\mathcal{F}$ on S, we have

$$H^i(S, \mathcal{F}) = H^i(S_{\text{ét}}, \mathcal{F}^{\text{ét}}).$$

In particular, from this and from Corollary 1.8, we obtain that for any field K

$$0 = \mathrm{Pic}(\operatorname{Spec} K) = H^1((\operatorname{Spec} K)_{\text{ét}}, \mathbb{G}_m) = H^1(\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}} | K), (\mathrm{K}^{\mathrm{sep}})^\times)$$

which is Hilbert's 90 theorem. Similarly, for all $i \geq 1$

$$0 = H^i(\operatorname{Spec} K, \mathcal{O}) = H^i((\operatorname{Spec} K)_{\text{ét}}, \mathbb{G}_a) = H^i(\mathrm{Gal}(\mathrm{K}^{\mathrm{sep}} | K), \mathrm{K}^{\mathrm{sep}}).$$

2 Kummer and Artin-Schreier sequences

Let $n \in \mathbb{N}$. In the past lectures we have introduced the functor μ_n

$$\begin{array}{rcl}\mathrm{Sch}^{opp} & \to & \mathrm{Ab} \\ S & \to & \{t \in \Gamma(S, \mathcal{O}_S) : t^n = 1\}\end{array}$$

which is representable and is therefore a sheaf for the fpqc topology, and in particular for the étale one.

Lemma 2.1 (Stacks,TAG 03PL)**.** *Let S be a scheme. If n is invertible on S then the Kummer sequence*

$$0 \to \mu_{n,S} \to \mathbb{G}_{m,S} \xrightarrow{(\cdot)^n} \mathbb{G}_{m,S} \to 0$$

is an exact sequence of sheaves both on $S_{\text{ét}}$ and $(\mathrm{Sch}/S)_{\text{ét}}$.

Proof. We just need to check surjectivity of $(\cdot)^n$. Let U be a scheme over S and $f \in \mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U^\times)$. We want to find an étale cover of U on which f is an n-th power. Let

$$V = \underline{\mathrm{Spec}}_U\, \mathcal{O}_U[T]/(T^n - f)) \xrightarrow{\pi} U$$

be given by the morphism $\mathcal{O}_U \to \mathcal{O}_U[T]/(T^n - f)$ of $\mathcal{O}_U$-algebras. The scheme V is built by gluing in the natural way the affine schemes $V_i := \mathrm{Spec}\, A_i[T]/(T^n - f_i)$ where $A_i = \Gamma(U_i, \mathcal{O}_U)$ for an affine cover $\{U_i\}_i$ of U, and $f_i = f_{|U_i}$. The ring map $A_i \to A_i[T]/(T^n - f_i)$ is finite and free, hence faithfully flat. Therefore the maps $V_i \to U_i$ are surjective, and π is surjective. Since each f_i is invertible, T^{n-1} is invertible in $A_i[T]/(T^n - f_i)$, and so is nT^{n-1}. Therefore each $V_i \to U_i$ is étale, and π is an étale cover. Moreover, $\pi^* f = T^n$, and this concludes the proof. □

Remark 2.2. The sequence above is not exact for the Zariski topology.

Remark 2.3. In the previous lecture, Raymond showed that for any $f \in \mathbb{G}_m(S)$, the fibre F_f of f via $(\cdot)^n$ is a $\mu_{n,S}$-torsor . He also showed that the set of $\mu_{n,S}$-torsors up to isomorphism is in bijection with $H^1(S_{\text{ét}}, \mu_n)$. The map $\mathbb{G}_{m,S}(S) = H^0(S_{\text{ét}}, \mathbb{G}_m) \to H^1(S_{\text{ét}}, \mu_n)$, $f \mapsto F_f$ is nothing else than the connecting morphism in the long exact sequence associated to the Kummer sequence.

Remark 2.4 (Stacks,TAG 040Q)**.** The abelian group $H^1(S_{\text{ét}}, \mu_n)$ has another explicit description. We can interpret it as the set of isomorphism classes of pairs $(\mathcal{L}, \alpha)$ with $\mathcal{L}$ an invertible sheaf on S and $\alpha : \mathcal{L}^{\otimes n} \xrightarrow{\cong} \mathcal{O}_S$ an isomorphism. In the long exact sequence of cohomology associated to the Kummer sequence

$$0 \to \mu_n(S) \to \mathbb{G}_m(S) \xrightarrow{(\cdot)^n} \mathbb{G}_m(S) \to H^1(S_{\text{ét}}, \mu_n) \to \mathrm{Pic}(S) \to \mathrm{Pic}(S)$$

the image of a pair $(\mathcal{L}, \alpha)$ in $\mathrm{Pic}(S)$ is the isomorphism class of $\mathcal{L}$, and the image of $a \in \mathbb{G}_m(S)$ inside $H^1(S_{\text{ét}}, \mu_n)$ is the class of the pair $(\mathcal{O}_S, a)$. To any such class $(\mathcal{L}, \alpha)$ we can associate a $\mu_{n,S}$-torsor, namely $\underline{Isom}_S((\mathcal{L}, \alpha), (\mathcal{O}_S, 1))$

Lemma 2.5 (Stacks,TAG 0A3J)**.** *Let S be a scheme in characteristic p. Then the Artin-Schreier sequence*

$$0 \to \underline{\mathbb{Z}/p\mathbb{Z}} \to \mathbb{G}_{a,S} \xrightarrow{x \mapsto x^p - x} \mathbb{G}_{a,S} \to 0$$

is an exact sequence of sheaves both on $S_{\text{ét}}$ and $(\mathrm{Sch}/S)_{\text{ét}}$.

Proof. Analogous to the one for the Kummer sequence. □

3 The pushforward functor

Let $f : X \to Y$ be a morphism of schemes. Then we define the functor

$$\begin{array}{rcl} f_* : \mathrm{Ab}(X_{\text{ét}}) & \to & \mathrm{Ab}(Y_{\text{ét}}) \\ \mathcal{F} & \mapsto & f_* \mathcal{F} : U \to \mathcal{F}(U \times_Y X) \end{array}$$

The functor f_* is left exact, and since the category $\mathrm{Ab}(X_{\text{ét}})$ has enough injectives we can define its right derived functors $R^p f_*$.

4 Étale cohomology of curves

We call a curve a separated, integral scheme of finite type over a field k of dimension 1.

We fix an algebraically closed field k and a smooth curve X over k. We denote $i_x : x \to X$ the inclusion of a closed point, $j : \eta \to X$ the inclusion of the generic point, and X^0 the set of closed points of X.

Theorem 4.1 (Stacks,TAG 03RI). *There is a short exact sequence of sheaves on $X_{\text{ét}}$*

$$0 \to \mathbb{G}_{m,X} \to j_* \mathbb{G}_{m,\eta} \to \bigoplus_{x \in X^0} i_{x*}\underline{\mathbb{Z}} \to 0$$

Proof. Let $U \to X$ be an étale morphism. Then U is the union of smooth, connected curves U_i and we can therefore assume that U is connected, and hence irreducible. Then there is an exact sequence

$$0 \to \Gamma(U, \mathcal{O}_U^\times) \to k(U)^\times \xrightarrow{div} \bigoplus_{y \in U_0} \mathbb{Z}.$$

which amounts to a sequence

$$0 \to \Gamma(U, \mathcal{O}_U^\times) \to \Gamma(\eta \times_X U, \mathcal{O}_{\eta \times_X U}^\times) \to \bigoplus_{x \in X^0} \Gamma(x \times_X U, \underline{\mathbb{Z}})$$

which is by definition

$$0 \to \mathbb{G}_m(U) \to j_* \mathbb{G}_{m,\eta}(U) \to (\bigoplus_{x \in X^0} i_{x*}\underline{\mathbb{Z}})(U).$$

Therefore it remains to check surjectivity for the sequence in the statement. For this, recall that if D is a divisor on U, then Zariski-locally D is the divisor of a rational function. □

Lemma 4.2 (Stacks,TAG 03RJ and TAG 03RK). *For any $p \geq 1$ we have*

i) $R^p j_* \mathbb{G}_{m,\eta} = 0$

ii) $H^p(X_{\text{ét}}, j_* \mathbb{G}_{m,\eta}) = 0$.

For a proof, see Stacks Project. The proof of i) relies on the fact that for an algebraically closed field k, an extension $k \hookrightarrow K$ of transcendence degree 1 and all $q \geq 1$, $H^q(\operatorname{Spec} K_{\text{ét}}, \mathbb{G}_m) = 0$ (TAG 03RK). The proof of ii) relies on i) and on the Leray spectral sequence (see the Appendix).

Lemma 4.3 (Stacks,TAG 03RL). *For all $q \geq 1$,*

$$H^q(X_{\text{ét}}, \bigoplus_{x \in X^0} i_{x*}\underline{\mathbb{Z}}) = 0.$$

Sketch of proof. One sees that for all $p > 0$, $R^p i_{x*}\mathbb{Z} = 0$ since i_x is a finite morphism (Stacks,TAG 03QP). Applying the Leray spectral sequence one finds $H^q(X_{\text{ét}}, i_{x*}\mathbb{Z}) = H^q(x_{\text{ét}}, \mathbb{Z})$ which vanishes because x is the spectrum of an algebraically closed field. □

Corollary 4.4 (Stacks,TAG 03RM). *Let X be a smooth curve over an algebraically closed field. Then for all $q \geq 2$*

$$H^q(X_{\text{ét}}, \mathbb{G}_m) = 0.$$

Proof. It is a consequence of the long exact sequence of cohomology associated to the exact sequence of Theorem 4.1. □

As a consequence of the lemmas above, we also get an exact sequence

$$0 \to \Gamma(X, \mathcal{O}_X^\times) \to \Gamma(X, j_* \mathbb{G}_{m,\eta}) \to \Gamma(X, \bigoplus_{x \in X^0} i_{x*}\mathbb{Z}) \to H^1(X_{\text{ét}}, \mathbb{G}_m) \to 0$$

which is the familiar

$$0 \to \Gamma(X, \mathcal{O}_X^\times) \to k(X)^\times \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 0$$

We now prove some results about cohomology of curves with finite coeffiecents.

Lemma 4.5 (Stacks,TAG 03RQ)**.** *Let X be a smooth projective curve of genus g over an algebraically closed field k and let $n \in \mathbb{N}$ be invertible in k. Then there are canonical identifications*

$$H^q(X_{\text{ét}}, \mu_n) = \begin{cases} \mu_n(k) & \text{if } q = 0 \\ \mathrm{Pic}^0(X)[n] & \text{if } q = 1 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } q = 2 \\ 0 & \text{if } q > 2. \end{cases}$$

Proof. We use the long exact sequence of cohomology coming from the Kummer sequence:

$$\begin{array}{l} 0 \longrightarrow \mu_n(k) \longrightarrow k^\times \xrightarrow{(\cdot)^n} k^\times \\ \hookrightarrow H^1(X_{et}, \mu_n) \longrightarrow \mathrm{Pic}(X) \xrightarrow{(\cdot)^n} \mathrm{Pic}(X) \\ \hookrightarrow H^2(X_{et}, \mu_n) \longrightarrow 0 \longrightarrow 0 \end{array}$$

From the vanishing of $H^q(X_{\text{ét}}, \mathbb{G}_m)$ when $q \geq 2$, we recover $H^q(X_{\text{ét}}, \mu_n) = 0$ for $q > 2$. Notice also that since k is algebraically closed, the map $k^\times \xrightarrow{(\cdot)^n} k^\times$ is surjective. Now consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow (\cdot)^n & & \downarrow (\cdot)^n & & \downarrow n & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

The abelian group $\mathrm{Pic}^0(X)$ can be identified with $\underline{\mathrm{Pic}}^0_{X/k}(k)$, that is, the set of k-rational points of the abelian variety $\underline{\mathrm{Pic}}^0_{X/k}$. Then the left vertical map in the diagram is surjective, since for any abelian variety A over an algebraically closed field k multiplication by $n \geq 1$ is a surjective morphism (see Mumford's book on Abelian Varieties, [Mumford]). Also, multiplication by n on $\mathbb{Z}$ is injective (this does not need a reference). Hence applying Snake's lemma to the diagram above, we complete the proof.

□

Over an algebraically closed field, we have a non-canonical isomorphism $\mu_n \cong \underline{\mathbb{Z}/n\mathbb{Z}}$. Moreover, since $\operatorname{char} k$ does not divide n, $\operatorname{Pic}^0[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ (see again Mumford's book). Hence we have non-canonical identifications

$$H^q(X_{\text{ét}}, \underline{\mathbb{Z}/n\mathbb{Z}}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } q = 0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & \text{if } q = 1 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } q = 2 \\ 0 & \text{if } q > 2. \end{cases}$$

Lemma 4.6 (Stacks,TAG 03RR). *Let X be an affine smooth curve over an algebraically closed field k and $n \in k^\times$. Then*

i) $H^0(X_{\text{ét}}, \mu_n) = \mu_n(k)$,

ii) $H^1(X_{\text{ét}}, \mu_n) \cong (\mathbb{Z}/n\mathbb{Z})^{2g+r-1}$ *where g is the genus of the smooth projective compactification $\overline{X}$ of X and r is the number of points in $\overline{X} \setminus X$,*

iii) $H^q(X_{\text{ét}}, \mu_n) = 0$ *for $q \geq 2$.*

Proof. We have $\operatorname{Div}(X) = \operatorname{Div}(\overline{X})/H$ where H is the subgroup of divisors supported on $\{x_1, \ldots, x_r\} = \overline{X} \setminus X$. Since $\operatorname{Div}(X) \to \operatorname{Pic}(X)$ is surjective, we have $\operatorname{Pic}(X) = \operatorname{Pic}(\overline{X})/R$, where R is the subgroup generated by the $\mathcal{O}_{\overline{X}}(x_i)$, $i = 1, \ldots, r$. Since $r \geq 1$, $\operatorname{Pic}^0(\overline{X}) \to \operatorname{Pic}(X)$ is surjective. Hence $\operatorname{Pic}(X)$ is also n-divisible. Using again the Kummer sequence, we get *i)* and *iii)*. For *ii)*, we use the interpretation of $H^1(X_{\text{ét}}, \mu_n)$ given in Remark 2.4 as the set of pairs $(\mathcal{L}, \alpha)$ up to isomorphism, with $\mathcal{L} \in \operatorname{Pic}(X)$ and $\alpha : \mathcal{L}^{\otimes n} \xrightarrow{\cong} \mathcal{O}_X$. There is an isomorphism

$$\{(\overline{\mathcal{L}}, D, \overline{\alpha})\}/R' \xrightarrow{\cong} \{(\mathcal{L}, \alpha)\}/\cong$$

where on the left hand side

- $\overline{\mathcal{L}} \in \operatorname{Pic}^0(\overline{X})$,
- $D \in H \subset \operatorname{Div}(\overline{X})$, i.e. D is supported on $\overline{X} \setminus X$,
- $\overline{\alpha} : \overline{\mathcal{L}}^{\otimes n} \xrightarrow{\cong} \mathcal{O}_{\overline{X}}(D)$
- R' is the subgroup of triples of the form $(\mathcal{O}_{\overline{X}}(D), nD, 1^{\otimes n})$.

Here the group structure on the set of triples is the obvious one. Notice also that any D appearing in such a triple is necessarily of degree zero, since $\mathcal{L}^{\otimes n} \in \operatorname{Pic}^0(\overline{X})$

The isomorphism above is given by restricting $\overline{\mathcal{L}}$ and $\overline{\alpha}$ to X. The reader can convince himself that this map is indeed an isomorphism.
Then we have an exact sequence

$$0 \to H^1(\overline{X}_{\text{ét}}, \mu_n) \to H^1(X_{\text{ét}}, \mu_n) \to \bigoplus_{i=1}^{r} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

where

- the leftmost map sends a pair $(\overline{\mathcal{L}}, \overline{\alpha})$ to $(\overline{\mathcal{L}}, 0, \overline{\alpha})$
- the central map sends $(\overline{\mathcal{L}}, D, \overline{\alpha})$ with $D = \sum_{i=1}^r a_i x_i$ to the r-tuple $(a_i)_{i=1}^r$

- the rightmost map is the sum.

Note that exactness at the second-last step follows from the fact that $\mathrm{Pic}^0(\overline{X})$ is n-divisible. Now, the kernel K of the map $\bigoplus_{i=1}^r \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is a free $\mathbb{Z}/n\mathbb{Z}$-module, hence the exact sequence of $\mathbb{Z}/n\mathbb{Z}$-modules

$$0 \to H^1(\overline{X}_{\text{ét}}, \mu_n) \to H^1(X_{\text{ét}}, \mu_n) \to K \to 0$$

splits. By lemma 4.5, $H^1(\overline{X}_{\text{ét}}, \mu_n)$ is free, hence $H^1(X_{\text{ét}}, \mu_n)$ must also be. Taking ranks in the exact sequence, the thesis follows. □

Appendix

Proposition (The Leray spectral sequence, Stacks,TAG 03QC)**.** *Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ an étale sheaf on X. Then there is a spectral sequence*

$$E_2^{p,q} = H^p(Y_{\text{ét}}, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X_{\text{ét}}, \mathcal{F}).$$

References

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