Statement of Lefschetz trace formula

Qijun Yan

November 7, 2014

Let A be a (left and right) noetherian ring throughout the notes unless otherwise stated.

1 Conventions and Basic Notions

Definition 1.1. [Sta, Tag 03RU] Let $\mathcal{F}$ be a sheaf of (left) A-modules.

1. We say $\mathcal{F}$ is the *constant sheaf with value* M if $\mathcal{F}$ is the sheafification of the presheaf $U \mapsto M$, where M is an A-module, denoted by $\underline{M}_X$ or $\underline{M}$.

2. We say $\mathcal{F}$ is a *constant sheaf* if it is isomorphic as a sheaf of A-modules to a sheaf as in 1.

3. We say $\mathcal{F}$ is *locally constant* if there exists a covering $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.

Definition 1.2. A sheaf of A-modules on $X_{\text{ét}}$ is *constructible* if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all i.

Here a subset Y of a topological space is said to be constructible if there is a finite disjoint decomposition $Y = \coprod_i (U_i \cap V_i^c)$, where U_i, V_i are open and retrocompact subsets of X ("retrocompact" means that the embeddings $U_i, V_i \hookrightarrow X$ are quasi-compact.)

We say a sheaf $\mathcal{F}$ of A-modules on X is *flat* if the stalks of F at all geometric points of X are flat A-modules. This is equivalent to saying that the functor $\mathcal{F} \otimes_A \_$ is exact in the category of sheaves of A-modules.

The following is a list of notations we will use in the sequel.

1. $\mathrm{Sh}(X, A)$ is the category of sheaves of A-modules on $X_{\text{ét}}$.

2. $C(X, A)$ is the category of complexes of sheaves of A-modules on $X_{\text{ét}}$.

3. $K(X, A)$ is the homotopy category of complexes of sheaves of A-modules on $X_{\text{ét}}$.

4. $D(X, A)$ is the derived category of complexes of sheaves of A-modules on $X_{\text{ét}}$.

For the notion of derived category we will give a further explaination in one of the following sections.

2 Some Functors on Sheaves

In this section we explain some functors and derived functors in étale cohomology.

Let $f : X \to Y$ be a morphism of schemes. We have seen the functor Γ of taking global sections and direct image functor f_*. They are both left exact functors, and we have also seen their right derived funtors $\mathbf{R}\Gamma$ and $\mathbf{R}f_*$ from $\mathrm{Sh}(X, A)$ to $\mathrm{Sh}(Y, A)$.

We define the inverse image functor $f^{-1} : \mathrm{Sh}(Y, A) \to \mathrm{Sh}(X, A)$ (also denoted by f^*, these two notations are the same in étale cohomology) is defined to the sheaf associated to the presheaf sending each $(U \to X)$ to

$$\mathrm{colim}_{(V,\phi)\in I}\mathcal{F}(V)$$

where I is such a category whose objects are pairs (V, ϕ), where $(V \to X)$ is an object of $Y_{\text{ét}}$ and $\phi : U \to V$ is a Y-morphism, and a morphism (V_1, ϕ_1) to (V_2, ϕ_2) is a Y-morphism $\psi : V_1 \to V_2$, such that $\phi_2 = \psi\phi_1$.

If Y is a quasi-compact quasi-separated scheme and f is separated of finite type, then we can define a funtor $f_! : \mathrm{Sh}(X, A) \to \mathrm{Sh}(Y, A)$ in the following way. For any sheaf $\mathcal{F}$ in $\mathrm{Sh}(X, A)$, and any object U in $Y_{\text{ét}}$, let

$$f_!\mathcal{F}(U) = \{s \in \mathcal{F}(U \times_Y X) : \text{the support of } s \text{ is proper over } U\}.$$

This is indeed a sheaf, see for example section 5.5 in [Fu11]. We have a canonical monomorphism $f_!\mathcal{F} \to f_*\mathcal{F}$. If f is proper we have $f_! = f_*$. The functor $f_!$ can also be defined in a different way when f is an étale morphism, see Section 5.5 in [Fu11], but it coincides with the defintion we give above.

Remark 2.1. If f is étale , then $f_!$ is exact and faithful, and for any morphism $g : Y' \to Y$, if we let $X' = X \times_Y Y'$ and $f' : X' \to Y'$, $g' : X' \to X$ the canonical projection, then we have $g^{-1}f_!\mathcal{F} = g'_!f'^{-1}\mathcal{F}$. See Proposition 5.5.1 in [Fu11] for the proof.

If $f : X \to Y$ is a closed immersion, then the functor f_* has a right adjoint (see section 5.4 in [Fu11]), which we will take as a definition of $f^! : \mathrm{Sh}(Y, A) \to \mathrm{Sh}(X, A)$, thanks to the uniqueness of adjoint functors. The functor $f^!$ is also left exact, so we have the derived functor $\mathbf{R}f^! : \mathrm{Sh}(Y, A) \to \mathrm{Sh}(X, A)$. We shall not use $f^!$ and $\mathbf{R}f^!$ in the formulation of Lefschetz trace formula.

3 Derived Categories

This section is mostly a complete copy from [Sta, Tag 03T3] in Stacks Project.
To set up notation, let $\mathcal{A}$ be an abelian category. Let $\mathrm{Comp}(\mathcal{A})$ be the abelian category of complexes in $\mathcal{A}$. Let $K(\mathcal{A})$ be the category of complexes up to homotopy, with objects equal to complexes in $\mathcal{A}$ and objects equal to homotopy classes of morphisms of complexes. This is not an abelian category. Loosely speaking, $D(A)$ is defined to be the category obtained by inverting all quasi-isomorphisms in $\mathrm{Comp}(\mathcal{A})$ or, equivalently, in $K(\mathcal{A})$. Moreover, we can define $\mathrm{Comp}^+(\mathcal{A}), K^+(\mathcal{A}), D^+(\mathcal{A})$ analogously using only bounded below complexes. Similarly, we can define $\mathrm{Comp}^-(\mathcal{A}), K^-(\mathcal{A}), D^-(\mathcal{A})$

using bounded above complexes, and we can define $\mathrm{Comp}^b(\mathcal{A}), K^b(\mathcal{A}), D^b(\mathcal{A})$ using bounded complexes.

Remark 3.1. Notes on derived categories.

1. There are some set-theoretical problems when $\mathcal{A}$ is somewhat arbitrary, which we will happily disregard.

2. The categories $K(A)$ and $D(A)$ may be endowed with the structure of triangulated category, but we will not need these structures in the following discussion.

3. The categories $\mathrm{Comp}(\mathcal{A})$ and $K(\mathcal{A})$ can also be defined when $\mathcal{A}$ is an additive category.

The homology functor $H^i : \mathrm{Comp}(\mathcal{A}) \to \mathcal{A}$ taking a complex $K^\bullet \mapsto H^i(K^\bullet)$ extends to functors $H^i : K(\mathcal{A}) \to \mathcal{A}$ and $H^i : D(\mathcal{A}) \to \mathcal{A}$.

Lemma 3.1. An object E of $D(\mathcal{A})$ is contained in $D^+(\mathcal{A})$ if and only if $H^i(E) = 0$ for all $i \ll 0$. Similar statements hold for D^- and D^+.

Proof. Hint: use truncation functors. See Derived Categories, Lemma **??**. □

Lemma 3.2. Morphisms between objects in the derived category.

1. Let $I^\bullet \in \mathrm{Comp}^+(\mathcal{A})$ with I^n injective for all $n \in \mathbf{Z}$. Then

$$\mathrm{Hom}_{D(\mathcal{A})}(K^\bullet, I^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(K^\bullet, I^\bullet).$$

2. Let $P^\bullet \in \mathrm{Comp}^-(\mathcal{A})$ with P^n is projective for all $n \in \mathbf{Z}$. Then

$$\mathrm{Hom}_{D(\mathcal{A})}(P^\bullet, K^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, K^\bullet).$$

3. If $\mathcal{A}$ has enough injectives and $\mathcal{I} \subset \mathcal{A}$ is the additive subcategory of injectives, then $D^+(\mathcal{A}) \cong K^+(\mathcal{I})$ (as triangulated categories).

4. If $\mathcal{A}$ has enough projectives and $\mathcal{P} \subset \mathcal{A}$ is the additive subcategory of projectives, then $D^-(\mathcal{A}) \cong K^-(\mathcal{P})$.

Proof. Omitted. □

Definition 3.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor and assume that $\mathcal{A}$ has enough injectives. We define the *total right derived functor of* F as the functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ fitting into the diagram

$$\begin{array}{ccc} D^+(\mathcal{A}) & \xrightarrow{RF} & D^+(\mathcal{B}) \\ \uparrow & & \uparrow \\ K^+(\mathcal{I}) & \xrightarrow{F} & K^+(\mathcal{B}). \end{array}$$

This is possible since the left vertical arrow is invertible by the previous lemma. If moreover, $\mathbf{R}F$ has finite cohomological dimension, then $\mathbf{R}F$ can be extended to a morphism from $D(A)$ to $D(B)$. Here we say $\mathbf{R}F$ has finite cohomological dimension if there exists an integer n such that $\mathbf{R}^iF(X) = 0$ for all $i > n$ and all objects X in A. In particular, if F is exact, then we have a total derived functor $RF : D(A) \to D(B)$.

Similarly, let $G : \mathcal{A} \to \mathcal{B}$ be a right exact functor and assume that $\mathcal{A}$ has enough projectives. We define the *total left derived functor of* G as the functor $LG : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ fitting into the diagram

$$\begin{array}{ccc} D^-(\mathcal{A}) & \xrightarrow{LG} & D^-(\mathcal{B}) \\ \uparrow & & \uparrow \\ K^-(\mathcal{P}) & \xrightarrow{G} & K^-(\mathcal{B}). \end{array}$$

This is possible since the left vertical arrow is invertible by the previous lemma.

Remark 3.2. From the definition of the total right derived functor and total left derived functor, we see that they both coincide with the usual right derived functors and the usual left derived funtors since an object in an abelian category A can be seen as an object in its derived category $D(A)$ (or D^+, or D^-, or D^b) in the obvious way. So we may just call a total right derived functor a right derived functor. Same for total left derived functor.

So far, for a morphism $f : X \to Y$ of schemes, we have the following derived functors.

1. $\mathbf{R}\Gamma : D^+(X, A) \to D^+(A)$, where $D^+(A)$ is the derived category of bounded below complexes of A-modules;
2. $\mathbf{R}f_* : D^+(X, A) \to D^+(Y, A)$;
3. $f^{-1} : D(Y, A) \to D(X, A)$;
4. $f_! : D(X, A) \to D(Y, A)$ if f is étale .

Remark 3.3. The functor f^{-1} is left adjoint to $\mathbf{R}f_*$.

We will define the total derived functors $\mathbf{R}f_!$ and $f^!$, after we have introduced the properly supported cohomology in the next section. To end up this section, we state the following useful lemma and apply it to some of the derived funtors we have defined above.

Lemma 3.3. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories, and $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ left exact functors. Assume $\mathcal{A}, \mathcal{B}$ have enough injectives. If $F(I)$ is right acyclic for G (i.e., $\mathbf{R}^iG(F(I)) = 0$, for all $i > 0$) for each injective object I in A, Then we have an isomorphism

$$\mathbf{R}(G \circ F) \xrightarrow{\simeq} \mathbf{R}G \circ \mathbf{R}F$$

for functors from $D^+(A)$ to $D^+(C)$.

Example 3.1. Let $f : X \to Y$ be a morphism of schemes. Then for any $K \in D^+(Y, A)$, there is a canonical map $\mathrm{H}^i(Y, K) \to \mathrm{H}^i(X, f^{-1}K)$ for each i. To see this, we apply the previous lemma to the following composition of functors

$$\mathrm{Sh}(X, A) \xrightarrow{f_*} \mathrm{Sh}(Y, A) \xrightarrow{\Gamma} \mathrm{Mod}(A),$$

where $\mathrm{Mod}(A)$ is the category of A-modules, and get an isomorphism

$$\mathbf{R}\Gamma \circ \mathbf{R}f_* = \mathbf{R}\Gamma.$$

Take values in $f^{-1}K$ on both sides of the equation, and then take the i-th cohomology we get an isomorphism

$$\mathrm{H}^i(Y, \mathbf{R}f_* f^{-1}\mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i(X, f^{-1}\mathcal{F}).$$

On the other hand, since f^{-1} is left adjoint to $\mathbf{R}f_*$, we have a cannonical morphism $K \to \mathbf{R}f_* f^{-1}K$, which induces a morphism on cohomologies

$$\mathrm{H}^i(X, K) \to \mathrm{H}^i(X, \mathbf{R}f_* f^{-1}K).$$

Now the composition of the two maps on cohomologies gives us the desired map $\mathrm{H}^i(Y, K) \to \mathrm{H}^i(X, f^{-1}K)$.

4 Cohomology with Proper Support

Let Y be a quasi-compact and quasi-separated scheme, and $f : X \to Y$ a separated morphism of finite type. A theorem of Nagata says that f admits a compactification, that is, there is a factorization $X \xrightarrow{j} \bar{X} \xrightarrow{\bar{f}} Y$ such that $f = \bar{f} \circ j$, where j is an open immersion and $\bar{f}$ is a proper morphism. Denote by $D^+_{\mathrm{tor}}(X, A)$ the subcategory of $D^+(X, A)$ consisting of complexes of torsion sheaves. Here a sheaf of A-modules is said to be a torsion sheaf if it is a *torsion sheaf* as an abelian sheaf (E.g. if A is a finite ring, then any sheaf of A-modules is a torsion sheaf). For any complex K in $D^+_{\mathrm{tor}}(X, A)$, set $\mathbf{R}f_! K = \mathbf{R}f_* f_! K$, then $\mathbf{R}f_! K$, up to isomorphism, is independent of the choice of the compactification of f, and we can define a derived functor $\mathbf{R}f_! : D^+_{\mathrm{tor}}(X, A) \to D^+_{\mathrm{tor}}(Y, A)$.

Remark 4.1.

1. The derived functor $\mathbf{R}f_!$ is not the derived functor of $f_!$.
2. The derived functor $\mathbf{R}f_!$ admits a right adjoint $f^! : D^+_{\mathrm{tor}}(Y, A) \to D^+_{\mathrm{tor}}$(X, A). Note that $f^! : \mathrm{Sh}(Y, A) \to \mathrm{Sh}(X, A)$ is only defined by closed immersion, and if f is indeed a closed immersion, $f^! : D^+_{\mathrm{tor}}(Y, A) \to D^+_{\mathrm{tor}}$ is the right derived functor of $f^! : \mathrm{Sh}(Y, A) \to \mathrm{Sh}(X, A)$. Unfortunately, they both share the same notation. Again, they are not needed for the formulation of Lefschetz trace formula.

Now let k be a separably closed field, and X a scheme separated of finite type over k. Let $X \xrightarrow{j} \bar{X} \to \mathrm{Spec} k$ be a compactification. For any complex K in $D^+_{\mathrm{tor}}(X, A)$, we define

$$\mathbf{R}\Gamma_c(X, K) = \Gamma(\mathrm{Spec} k, \mathbf{R}f_* K) = \mathbf{R}\Gamma(\bar{X}, j_! K)$$

$$\mathrm{H}^i_c(X,K)=\mathrm{H}^i(\mathbf{R}\Gamma_c(X,K))=\mathrm{H}^i(\bar{X},j_!K)$$

For any sheaf $\mathcal{F}$ on $X_{\text{ét}}$, we define

$$\Gamma_c(X,\mathcal{F})=\mathrm{H}^0_c(X,\mathcal{F})=\Gamma(\bar{X},j_!\mathcal{F}).$$

Remark 4.2. The module $\Gamma_c(X,\mathcal{F})$ is a submodule of $\Gamma(X,\mathcal{F})$ consisting of those sections in $\Gamma(X,\mathcal{F})$ with support proper over Spec$k$.

5 Traces and Perfect Complexes

Let A be a ring as in the begining of the notes. We let $A^\natural$ be the quotient of the additive group $(A,+)$ by the subgroup generated by elements of the form $ab-ba$, $a,b\in A$.

Example 5.1. Let $A=\mathbb{Z}/\ell[G]$, where G is a finite group. Then we have

$$A^\natural=\bigoplus_{\text{conjugacy class of } G}\mathbb{Z}/\ell$$

Let $f:A^{\oplus n}\to A^{\oplus n}$ be an endomorphism of free A-modules. And let f be represented by a matrix (a_{ij}) with respect to a basis, we define the trace $\mathrm{Tr}(f)$ to be the image of $\sum_{i=1}^n a_{ii}$ in $A^\natural$.

Example 5.2. Let $f:A^{\oplus r}\to A^{\oplus s}$, $g:A^{\oplus s}\to A^{\oplus r}$ be two homomorphisms of free A-modules. Then we have $\mathrm{Tr}(fg)=\mathrm{Tr}(gf)$. One can check this easily.

Let P be a finitely generated projective A-module, and $f:P\to P$ an endomorphism of A-modules. Let $p:A^{\oplus n}\to P$ be a surjective homomorphism of A-modules, then p has a split $i:P\to A^{\oplus n}$. We define the trace $\mathrm{Tr}(f)=\mathrm{Tr}(i\circ f\circ p)$. One may use the previous example to show that the trace $\mathrm{Tr}(f)$ is independent of the surjection p we choose. Moreover one can check that if $f:P\to Q$, $g:Q\to P$ are two homomorphisms of finitely generated projective A-modules, then we have $\mathrm{Tr}(fg)=\mathrm{Tr}(gf)$.

Let $P^\cdot$ be a bounded complex of finitely generated projective A-modules, and let $f=(f^i):P^\cdot\to P^\cdot$ be a morphism of complexes. We define

$$\mathrm{Tr}(f)=\sum_i(-1)^i\mathrm{Tr}(f^i)$$

If f is homotopic to 0, then $\mathrm{Tr}(f)=0$. Indeed, if $h^i:P^i\to P^{i-1}$ are homomorphisms such that $h^{i+1}d^i+d^{i-1}h^i=f^i$, then we have

$$\begin{aligned}\mathrm{Tr}(f) &= \sum_i(-1)^i\mathrm{Tr}(h^{i+1}d^i+d^{i-1}h^i)\\ &= \sum_i(-1)^i\mathrm{Tr}(h^{i+1}d^i)+\sum_i(-1)^i\mathrm{Tr}(d^{i-1}h^i)\\ &= \sum_i(-1)^i\mathrm{Tr}(d^ih^{i+1})+\sum_i(-1)^i\mathrm{Tr}(d^{i-1}h^i)\\ &= 0.\end{aligned}$$

A bounded complex of finitely generated projective A-modules is called *perfect*. Let $K^b_{\mathrm{perf}}(A)$ be the category of perfect complexes up to homotopy. By Lemma 3.2, we see that the canonical functor $K^b_{\mathrm{perf}}(A) \to D^b(A)$ is faithful, where $D^b(A)$ is the derived category of bounded complexes of A-modules. Let $D^b_{\mathrm{perf}}(A)$ be the essential image of this functor. By the discussion above, one sees that for each complex K in $D^b_{\mathrm{perf}}(A)$, and each endomorphism $f : K \to K$, we can define the trace $\mathrm{Tr}(f, K) = \mathrm{Tr}(f)$.

6 Lefschetz Trace Formula

Let X be a scheme of characteristic $p > 0$. Then we have the Frobenius $\mathrm{Fr}_X : X \to X$ given by sending a local section x to x^q with q a power of p. And for any X-scheme U, we can form the relative Frobenius map $\mathrm{Fr}_{X/U} : U \to U^{(q)}$, where

$$U^{(q)} = U \times_{X,\mathrm{Fr}_X} X.$$

One sees easily that the formulation of the relative Frobenius $\mathrm{Fr}_{X/U}$ is functorial in U. One important result (though not hard to prove) here is that if U is étale over X, then the relative Frobenius $\mathrm{Fr}_{X/U}$ is an isomorphism.

Remark 6.1. The requirement that $\mathrm{Fr}_{X/U}$ is an isomorphism is equivalent to the requirement that the following commutative diagram is cartesian.

$$\begin{array}{ccc} U & \xrightarrow{\mathrm{Fr}_U} & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{\mathrm{Fr}_X} & X \end{array}$$

Lemma 6.1. If X is a scheme of characteristic $p > 0$, then for any sheaf in $\mathcal{F}$ in $\mathrm{Sh}(X, A)$, there is an isomorphism $\mathcal{F} \simeq \mathrm{Fr}_{X*}\mathcal{F}$.

Proof. For any etale morphism $U \to X$ in $X_{\text{ét}}$, the relatve Frobenius map $\mathrm{Fr}_{U/X}$ gives an isomorphism of A-modules

$$\mathcal{F}(U) \to \mathrm{Fr}_{X*}\mathcal{F}(U) = \mathcal{F}(U^{(q)}).$$

Since the relative Frobenius map is functorial in U, we indeed get an isomorphism $\mathcal{F} \simeq \mathrm{Fr}_{X*}\mathcal{F}$. □

By adjunction, we also have a morphism $\mathrm{Fr}^*_{\mathcal{F}} : \mathrm{Fr}_X^{-1}\mathcal{F} \to \mathcal{F}$. One can show that this is also an isomorphism. Similarly for any complex K in $D(X, A)$, we also have a morphism $\mathrm{Fr}^*_K : \mathrm{Fr}_X^{-1}K \to K$.

To finally go to the setting of Lefschetz trace formula, we still need the notion of "finite tor-dimension". For a scheme X, we can define a functor

$$- \overset{L}{\otimes} - : D^-(X, A) \times D^-(X, A) \to D^-(X, A)$$

given in the following way (see section 6.4 in [Fu11]). For any two complexes M, N in $D^-(X, A)$, let P (resp. Q) be a flat resolution of M (resp. N), then we define

$$- \overset{L}{\otimes} -(M, N) = P \otimes_A Q.$$

We denote $- \overset{L}{\otimes} -(M, N)$ by $M \overset{L}{\otimes} N$. For any complex K in $D^b(X, A)$, we say K has *finite Tor-dimension* if there exists an integer i such that $\mathrm{H}^i(K \overset{L}{\otimes} \underline{E}) = 0$ for any $i < n$, and any A-module E, where $\underline{E}$ is the constant sheaf with value E.

From now on we go to the setting of Lefschetz trace formula. Let $k = \mathbb{F}_q$ be a finite field of characteristic $p > 0$, A a noetherian $\mathbb{Z}/\ell^n$-algebra with $(l, p) = 1$, X_0 a scheme separated and of finite type over k, and K_0 a complex in $D^b_{\text{ctf}}(X_0, A)$. Let $X_0 \to \overline{X_0}$ be an compactification of X_0. By the discussion above, for the Frobenius morphism $\mathrm{Fr}_{X_0} : X_0 \to X_0$, there is a canonical morphism $\mathrm{Fr}^*_{K_0} : \mathrm{Fr}^{-1}_{X_0} K_0 \to K_0$. Let $\bar{k}$ be an algebraic closure of k, and denote by X, $\overline{X}$, $j : X \hookrightarrow \overline{X}$, K, $\mathrm{Fr} : X \to X$, $\mathrm{Fr}^*_{K_0} : \mathrm{Fr}^{-1} K_0 \to K_0$ the pull back of X_0, $\overline{X_0}$, $X_0 \hookrightarrow \overline{X_0}$, K_0, $\mathrm{Fr}_{X_0} : X_0 \to X_0$, and $F^*_{K_0} : \mathrm{Fr}^{-1}_{X_0} K_0 \to K_0$ along the morphism $\mathrm{Spec}\bar{k} \to \mathrm{Spec}k$. Now, in order to give the Lefschetz trace formula, we firstly define the so called "global Lefschetz number". For this, we want to set up an endomorphism

$$\mathbf{R}\Gamma_c(X, K) \to \mathbf{R}\Gamma_c(X, K).$$

To do this, notice that we have the following cartesian diagram (see Remark 6.1)

$$\begin{array}{ccc} X_0 & \xrightarrow{\mathrm{Fr}_{X_0}} & X_0 \\ \big\downarrow & & \big\downarrow \\ \overline{X_0} & \xrightarrow{\mathrm{Fr}_{\overline{X_0}}} & \overline{X_0} \end{array}$$

By base change along $\mathrm{Spec}\bar{k} \to \mathrm{Spec}k$, we get a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \big\downarrow j & & \big\downarrow j \\ \bar{X} & \xrightarrow{F'} & \bar{X} \end{array}$$

By Remark 2.1 we have

$$F'^{-1} j_! K = j_! F^{-1} K$$

Now denote by $F^* : \mathbf{R}\Gamma_c(X, K) \to \mathbf{R}\Gamma_c(X, K)$ the composite

$$\begin{array}{rcl} \mathbf{R}\Gamma_c(X, K) & \simeq & \mathbf{R}\Gamma(\bar{X}, j_! K) \\ & \xrightarrow{(*)} & \mathbf{R}\Gamma(\bar{X}, F'^{-1} j_! K) \\ & \simeq & \mathbf{R}\Gamma(\bar{X}, j_! F^{-1} K) \\ & \xrightarrow{F^*_{K_0}} & \mathbf{R}\Gamma(\bar{X}, j_! K) \\ & = & \mathbf{R}\Gamma_c(X, K), \end{array}$$

where $(*)$ is due to Exmaple 3.1.

Proposition 6.1. The complex $\mathbf{R}\Gamma_c(X, K)$ above lies in $D^b_{\text{perf}}(A)$.

Hence we can define the trace $\text{Tr}(F^*, \mathbf{R}\Gamma_c(X, K))$ of F^*, called the *global Lefschetz number* of $K_0 \in D^b_{\text{ctf}}(X_0, A)$.

We need still to define the so called "local Lefschetz number" in the followin way. For any rational point $x \in X_0(k)$, and $\text{Spec}\bar{k} \to X$, the geometric point lying over x, $K_{\bar{x}} = \bar{x}^{-1}K_0$ is a perfect complex, i.e., we have $K_{\bar{x}} \in D^b_{\text{perf}}(A)$. Let $F_x : \text{Spec}\bar{k} \to \text{Spec}\bar{k}$ be the morphism induced by $k \to k$ sending $x \in k$ to $x^{1/q}$. As in Lemma 6.1, we also have an isomorphism $K_{\bar{x}} \to F_{x*}K_{\bar{x}}$, which by adjunction induces a morphism $F_x^{-1}K_{\bar{x}} \to K_{\bar{x}}$. Then the composite $K_{\bar{x}} \to F_x^{-1}K_{\bar{x}} \to K_{\bar{x}}$ gives an endomorphism $F_x^* : K_{\bar{x}} \to K_{\bar{x}}$. The sum

$$\sum_{x \in X_0(k)} \text{Tr}(F_x^*, K_{\bar{x}})$$

is the so called local "Lefschetz trace number" of $K_0 \in D^b_{\text{perf}}(X_0, A)$.

Finally we can state the Lefschetz trace formula, which says that the global Lefschetz number is equal to the local Lefschetz number.

Theorem 6.1. (Lefschetz Trace Formula) Notations as above, we have

$$\sum_{x \in X_0(k)} \text{Tr}(F_x^*, K_{\bar{x}}) = \text{Tr}(F^*, \mathbf{R}\Gamma_c(X, K)).$$

References

[Fu11] L. Fu. *Etale Cohomology Theory.* Nankai tracts in mathematics. World Scientific Publishing Company, Incorporated, 2011.

[Sta] The Stacks Project Authors. *stacks project.* *http://stacks.math.columbia.edu.*