THE ÉTALE SITE

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Date: March 29, 2011.

1. Smooth morphisms

In these first two sections, we will not prove the statements we give. Refer to [S] 35.26. to see where you can find the proofs.

First, let us define smooth ring maps. We do this in 3 steps, cf. [S] 35.26.

Definition 1.1. Let d be a non-negative integer.

(1) Let k be an algebraically closed field, and let R be a k-algebra. Then R is *smooth of relative dimension* d if it is of finite type, its dimension is d, and the module $\Omega_{R/k}$ of differentials is a finitely generated locally free R-module of rank d.
(2) Let k be an arbitrary field, and $\overline{k}$ its algebraic closure. Let R be a k-algebra. Then R is *smooth of relative dimension* d if $R \otimes_k \overline{k}$ is smooth of relative dimension d over $\overline{k}$.
(3) Let $f: R \to S$ be a ring map. Then f is *smooth of relative dimension* d if it is flat, finitely presented, and for all primes $\mathfrak{p}$ of R, the fibre ring $k(\mathfrak{p}) \otimes_R S$ is smooth of relative dimension d over $k(\mathfrak{p})$.

Now we can define smooth morphisms of rings.

Definition 1.2. Let d be a non-negative integer.

(1) Let k be an algebraically closed field, and let X be a k-scheme. Then X is *smooth of relative dimension* d if it is locally of finite type, and each affine open is the spectrum of some smooth k-algebra of relative dimension d.
(2) Let k be an arbitrary field, and $\overline{k}$ its algebraic closure. Let X be a k-scheme. Then X is *smooth of relative dimension* d if $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is smooth of relative dimension d over $\overline{k}$.
(3) Let $f: X \to S$ be a morphism of schemes. Then f is *smooth of relative dimension* d if it is flat, locally of finite presentation, and all fibres are smooth of relative dimension d.

We can also define smoothness in a different way, with a more differential-geometric flavor.

Definition 1.3. Let R be a ring, and $n \geq d$ be non-negative integers. Let $f_1, \ldots, f_{n-d} \in R[x_1, \ldots, x_n]$. Then the canonical map $R \to S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-d})$ is called a *standard smooth ring map of relative dimension* d, if the polynomial

$$g = \det(\tfrac{\partial f_j}{\partial x_i})_{i,j=1}^{n-d}$$

is invertible in S. The morphism of schemes $\operatorname{Spec} S \to \operatorname{Spec} R$ induced by this map is called a *standard smooth morphism of relative dimension* d.

These two notions of smoothness coincide:

Proposition 1.4. *Let $f: X \to S$ be a morphism of schemes. Then f is smooth of relative dimension d if and only if for all $x \in X$, there exist affine opens $U \subseteq X$ and $V \subseteq S$ such that $x \in U$, $f(U) \subseteq V$, and $f|_U : U \to V$ is standard smooth of relative dimension d.*

We have the following properties of standard smooth ring maps.

Proposition 1.5.

(1) *A composition of standard smooth ring maps (of relative dimensions d and d') is again standard smooth (of relative dimension $d + d'$);*
(2) *A base change of a standard smooth ring map (of relative dimension d) is again standard smooth (of relative dimension d);*

(3) *Tensor products of standard smooth ring maps (of relative dimensions d and d') are again standard smooth (of relative dimension $d + d'$).*

Proof. Straightforward. □

It automatically follows that compositions, base changes and fibre products of smooth morphisms of schemes are again smooth.

Proposition 1.6. *Smooth morphisms are open.*

2. Étale morphisms

Definition 2.1. Let $f\colon R \to S$ be a ring map. Then f is called *étale* if it is a smooth ring map of relative dimension 0.

Definition 2.2. Let $f\colon X \to S$ be a morphism of schemes. Then f is called *étale* if it is smooth of relative dimension 0.

Then from the previous section it automatically follows that:

(1) Compositions of étale morphisms are étale.
(2) Base changes of an étale morphism are étale.
(3) Fibre products of étale morphisms are étale.
(4) Étale morphisms are open.
(5) A morphism of schemes $f\colon X \to S$ is étale if and only if f is flat, locally of finite presentation, and its fibres are étale.

So we would like to characterize the étale schemes over a field k.

Proposition 2.3. *Let $f : X \to \operatorname{Spec} k$ be a morphism of schemes. Then f is étale if and only if X is a disjoint union of spectra of finite separable field extensions of k.*

We now list some other useful properties of étale morphisms.

Proposition 2.4.

(1) *A ring map is étale if and only if it is standard smooth of relative dimension 0.*
(2) *If $X \to S$ is étale, and S is reduced, then so is X.*
(3) *If $X \to S$ and $Y \to S$ are étale, then any S-morphism $X \to Y$ is also étale.*

Definition 2.5. Let R be a ring, and let $f, g \in R[t]$. Then the canonical map $R \to S = (R[t]/(f))_g$ is called a *standard étale ring map* if f is monic and the derivative of f is invertible in S, and the induced morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ is called a *standard étale morphism.*

Note that standard étale ring maps are indeed étale. But while base changes of standard étale maps are again standard étale, the same does not hold for compositions of standard étale maps. It also turns out that morphisms of schemes are étale if and only if they are locally standard étale. See [S], Lemma 7.125.15 for the proof.

We now give a useful characterization of étale morphisms between two non-singular varieties over an algebraically closed field.

Proposition 2.6. *Let k be an algebraically closed field, and let $f\colon X \to S$ be a morphism of non-singular k-varieties. Then f is étale if and only if for every $x \in X$, f induces an isomorphism $T_x(X) \to T_{f(x)}(S)$.*

We give a couple of examples of étale morphisms.

Example 2.7. (1) Open immersions are étale.

(2) Let k be an algebraically closed field, and let $f \in k[x_n]$ be separable. Consider $V(f) \subseteq \mathbb{A}^n_k$, and the projection map $V(f) \to \mathbb{A}^{n-1}_k$ on the first $(n-1)$ coordinates. This map is induced by the canonical ring map $k[x_1, \dots, x_{n-1}] \to k[x_1, \dots, x_n]/(f)$, which is standard étale. Hence the projection map is étale.

(3) On the other hand, if k is an algebraically closed field of characteristic not equal to 2, and $X = V(y^2 - x) \subseteq \mathbb{A}^2_k$, then the projection map $X \to \mathbb{A}^1_k$ on the first coordinate is *not* étale, since the fibre X_0 is not reduced, while $\operatorname{Spec} k$ is reduced (using Proposition 2.3). But if we consider the map $X - \{0\} \to \mathbb{A}^2_k$ instead, then this map *is* étale, since it is induced by the standard étale ring map $k[x] \to (k[x,y]/(y^2 - x))_y$.

(4) Let k be a field, and consider the morphism $\phi\colon \coprod_{x \in \mathbb{A}^1_k} \operatorname{Spec} \mathcal{O}_{\mathbb{A}^1_k, x} \to \mathbb{A}^1_k$. As shown before (by René), this is not an fpqc morphism, so, as we'll see later, that already implies that ϕ is not étale, but we can also see this by observing that its fibres aren't discrete, and then using Proposition 2.3.

(5) As we've seen in Proposition 2.3, if k is a field, and for i in some index set I, k_i is a separable field extension, then $\coprod \operatorname{Spec} k_i \to \operatorname{Spec} k$ is étale.

(6) Let $K \subseteq L$ be an extension of number fields, and let $\mathcal{O}_K, \mathcal{O}_L$ be their rings of integers. Let Δ be the discriminant of $K \subseteq L$. Then the morphism $\operatorname{Spec} \mathcal{O}_L[\Delta^{-1}] \to \operatorname{Spec} \mathcal{O}_K$ is étale; it is flat, locally of finite presentation, and all fibres are étale.

(7) Let k be a field. Then note that $\mathbb{G}_{m,k} = \operatorname{Spec} k[t, t^{-1}]$, and that the n-th power map $\cdot^n\colon \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ is induced by the ring map $k[t, t^{-1}] \to k[t, t^{-1}]$ given by $t \mapsto t^n$. Since $k[t, t^{-1}] = k[t^n, t^{-n}][u]/(u^n - t^n)$, and both n and u^{n-1} are units, it follows that this ring map is standard étale. Hence the n-th power map is étale.

(8) Let E be an elliptic curve over an algebraically closed field, and let $n \in \mathbb{Z}$ be invertible in k. Then the multiplication-by-n map $n\cdot\colon E \to E$ is étale;

Note that the addition map $+\colon E \times E \to E$ is the identity on each of the components, so the same follows for the induced map on tangent spaces $+^*\colon T_0(E) \times T_0(E) \to T_0(E)$. Hence $+^*$ is also the addition map on $T_0(E)$. We deduce that the map $T_0(E) \to T_0(E)$ induced by the multiplication-by-n map on E is the multiplication-by-n map on $T_0(E)$. Since n is invertible in k, it follows that this map is an isomorphism. Now, by considering, for all $P \in E$, the commutative diagrams (and noting that the vertical maps are isomorphisms)

$$\begin{array}{ccc} E & \xrightarrow{n\cdot} & E \\ {\scriptstyle P+}\downarrow & & \downarrow{\scriptstyle nP+} \\ E & \xrightarrow[n\cdot]{} & E \end{array} \qquad \begin{array}{ccc} T_0(E) & \xrightarrow{n\cdot^*} & T_0(E) \\ \downarrow & & \downarrow \\ T_P(E) & \xrightarrow[n\cdot^*]{} & T_{nP}(E) \end{array}$$

it follows that by Proposition 2.6, $n\cdot$ is indeed étale.

Now assume that k is a perfect field, not necessarily algebraically closed. Since $k \to \overline{k}$ is faithfully flat, for all maximal ideals of a k-algebra B, the sequence $0 \to \mathfrak{m} \otimes_k \overline{k} \to B \otimes_k \overline{k} \to (B/\mathfrak{m}) \otimes_k \overline{k}$ is exact, implying that $B^\times = B \cap (B \otimes_k \overline{k})mg$. Hence it follows that if the map of $\overline{k}$-algebras $A \otimes_k \overline{k} \to B \otimes_k \overline{k}$ is étale, then the map $A \to B$ is étale as well. We deduce the multiplication-by-n map $E \to E$ is étale even if k is not algebraically closed.

3. The étale site

Definition 3.1. An *étale covering* of a scheme U is a family of morphisms of schemes $\{\phi_i \colon U_i \to U\}$ such that each ϕ_i is étale, and the $\phi_i(U_i)$ cover U.

Example 3.2.

(1) If $f \colon X \to S$ is étale, then $\{f\}$ is an étale covering.
(2) Every Zariski covering is an étale covering.

Lemma 3.3. *Every étale covering is an fpqc covering.*

To prove this, we need the following proposition, mentioned by René before.

Proposition 3.4. *Let $\mathcal{U} = \{\phi_i \colon U_i \to U\}$ be a family of morphisms of schemes. Then $\mathcal{U}$ is an fpqc covering if and only if all ϕ_i are flat, jointly surjective, and for all affine opens $V \subseteq U$, there is a finite number of affine opens $V_{ij} \subseteq U_i$, the union of whose images is V.*

Proof of Lemma. Let $\mathcal{U} = \{\phi_i \colon U_i \to U\}$ be an étale covering. Then by definition, the ϕ_i are flat and jointly surjective.

So now suppose that $V \subseteq U$ is an affine open subset, and write $\phi_i^{-1}(V)$ as a union $\bigcup_{j_i} V_{ij} \subseteq U_i$ of affine open subsets. Since all the ϕ_i are étale, hence open, it follows that $\mathcal{V} = \{\phi_i(V_{ij})\}$ is an open cover of V. Since V is affine, hence quasi-compact, it follows that $\mathcal{V}$ has a finite subcover. Hence Proposition 3.4 implies that $\mathcal{U}$ is an fpqc covering. □

So anything that is true for fpqc coverings, remains true for étale coverings.

We now define the étale sites over a scheme S.

Definition 3.5. The *big étale site over S*, denoted $(\mathrm{Sch}_S)_{\text{ét}}$ is the site of which the underlying category is that of Sch_S, and where the coverings are the étale coverings.

The *small étale site over S*, denoted $S_{\text{ét}}$ is the site of which the underlying category is the full subcategory of Sch_S consisting of all the schemes that are étale over S, and where the coverings are the étale coverings.

So now Lemma 3.3 also implies that fpqc sheaves are étale as well.

Note that by Proposition 2.4, $\{\phi_i \colon U_i \to U\}$ is a covering in the small étale site over S if and only if the ϕ_i are jointly surjective.

Also note that $S_{\text{ét}}$ is a full category of $(\mathrm{Sch}_S)_{\text{ét}}$, with the induced topology. So the restriction functor $(\mathrm{Sch}_S)_{\text{ét}} \to S_{\text{ét}}$ is exact and maps injectives to injectives. Hence we have the following:

Proposition 3.6. *Let S be a scheme, and let $\mathcal{F}$ be a sheaf of abelian groups on $(\mathrm{Sch}_S)_{\text{ét}}$. Then $\mathcal{F}|_{S_{\text{ét}}}$ is a sheaf on $S_{\text{ét}}$, and for all $p \geq 0$, we have*

$$\mathrm{H}^p(S, \mathcal{F}|_{S_{\text{ét}}}) = \mathrm{H}^p(S, \mathcal{F}).$$

Hence there is no chance of confusion when we denote this cohomology group by $\mathrm{H}^p_{\text{ét}}(S, \mathcal{F})$.

4. Kummer theory

Our goal in this section is to study the map $\Gamma(S,\mathcal{O}_S^\times) \xrightarrow{\cdot n} \Gamma(S,\mathcal{O}_S^\times)$.

Let $n \in \mathbb{N}$, and consider the presheaf μ_n defined by

$$\mathrm{Sch}^{\mathrm{opp}} \to \mathrm{Ab}\colon S \mapsto \{t \in \Gamma(S,\mathcal{O}_S^\times) \mid t^n = 1\}.$$

This is a functor which is represented by $\mu_n' = \operatorname{Spec}\mathbb{Z}[t]/(t^n-1)$. It follows that the presheaf μ_n is a sheaf for the fpqc topology (so for the étale topology as well).

Lemma 4.1. *If $n \in \mathcal{O}_S^\times$, then*

$$0 \longrightarrow \mu_{n,S} \longrightarrow \mathbb{G}_{n,S} \xrightarrow{\cdot n} \mathbb{G}_{n,S} \longrightarrow 0$$

is a short exact sequence of sheaves on both the big and small étale sites of S.

Remark. This statement is not true for the Zariski topology on S.

Proof. By definition, $\mu_{n,S}$ is the kernel of the n-th power map $\mathbb{G}_{n,S} \to \mathbb{G}_{n,S}$. So we're done if we show that the n-th power map is surjective, i.e. for any scheme U over S, and any element $f \in \Gamma(U,\mathcal{O}_U^\times)$, there is an étale covering $\{U_i \to U\}$ such that for all i, the restriction $f|_{U_i}$ is an n-th power. To this end, consider the morphism of schemes

$$\pi\colon \operatorname{Spec}_U(\mathcal{O}_U[T]/(T^n - f)) \to U.$$

Remark. The *relative spectrum* $\operatorname{Spec}_S \mathcal{A}$ of a quasi-coherent $\mathcal{O}_S$-algebra $\mathcal{A}$ over a scheme S is the unique S-scheme satisfying:

(1) for all affine open $U \subseteq S$ there exists an isomorphism $i_U\colon \pi^{-1}(U) \to \operatorname{Spec}\mathcal{A}(U)$;
(2) for all affine opens $U \subseteq U' \subseteq S$, the map $\operatorname{Spec}\mathcal{A}(U) \to \operatorname{Spec}\mathcal{A}(U')$ induced by the sheaf structure on $\mathcal{A}$, is given by the composition $i_{U'} \circ \iota \circ i_U^{-1}$, where $\iota\colon \pi^{-1}(U) \to \pi^{-1}(U')$ is the inclusion.

Let $\operatorname{Spec} A \subseteq U$ be an affine open subset, and let $a \in A^\times$ be the unit corresponding to $f|_{U'}$. Then note that $\pi^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$, where $B = A[T]/(T^n - a)$. Since the ring map $A \to B$ is an integral extension of A, it follows that the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective. Also note that T^{n-1} is invertible in B, and, by our assumption on n, n is invertible in B as well. We deduce that the ring map $A \to B$ is standard étale. So the map $\pi|_{\operatorname{Spec} B}\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is surjective and étale, for every affine open subset $\operatorname{Spec} A \subseteq U$. It follows that π is surjective and étale as well. Hence $\{\pi\}$ is an étale covering. Also note that now, for the class t of $T \in \Gamma(U',\mathcal{O}_{U'}^\times)$, we have $f|_{U'} = t^n$, so the cover $\{\pi\}$ satisfies the desired properties. □

From this lemma (and the fact that $\mathrm{H}^1_{\text{ét}}(S, \mathbb{G}_{m,S}) = \operatorname{Pic} S$) it follows that we have a long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{H}^0_{\text{ét}}(S, \mu_{n,S}) \longrightarrow \Gamma(S, \mathcal{O}_S^\times) \xrightarrow{.n} \Gamma(S, \mathcal{O}_S^\times) \\ \xrightarrow{\delta} \mathrm{H}^1_{\text{ét}}(S, \mu_{n,S}) \longrightarrow \operatorname{Pic} S \xrightarrow{.n} \operatorname{Pic} S \\ \longrightarrow \mathrm{H}^2_{\text{ét}}(S, \mu_{n,S}) \longrightarrow \dots, \end{array}$$

if n is invertible on S. So to understand the n-th power map $\Gamma(S, \mathcal{O}_S^\times) \to \Gamma(S, \mathcal{O}_S^\times)$ better, we take a better look at $\mathrm{H}^1_{\text{ét}}(S, \mu_{n,S})$. It turns out there is a more direct description of this cohomology group.

Let S be a scheme, and let $n \in \mathbb{N}$ be invertible on S. Consider the set $\mathcal{I}$ of pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is an invertible sheaf on S, and $\alpha\colon \mathcal{L}^{\otimes n} \to \mathcal{O}_S$ is a trivialization. We call two such pairs $(\mathcal{L}, \alpha)$ and $(\mathcal{L}, \alpha')$ *isomorphic* if there exist an isomorphism $\phi\colon \mathcal{L} \to \mathcal{L}'$ of invertible sheaves making the following diagram commute.

$$\begin{array}{ccc} \mathcal{L}^{\otimes n} & \xrightarrow{\alpha} & \mathcal{O}_S \\ \phi^{\otimes n} \downarrow & & \downarrow 1 \\ \mathcal{L}'^{\otimes n} & \xrightarrow{\alpha'} & \mathcal{O}_S \end{array}$$

Note that

(1) $$\operatorname{Iso}_S((\mathcal{L}, \alpha), (\mathcal{L}', \alpha')) = \begin{cases} \emptyset & \text{if not isomorphic} \\ \mathrm{H}^0(S, \mu_{n,S}) \cdot \phi & \text{if } \phi \text{ is an isomorphism of pairs} \end{cases}$$

Let $\mathcal{J}$ be the set of isomorphism classes in $\mathcal{I}$.

Note that for any two $(\mathcal{L}, \alpha), (\mathcal{L}', \alpha') \in \mathcal{I}$, their tensor product $(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha') = (\mathcal{L} \otimes \mathcal{L}', \alpha \otimes \alpha')$ is again in $\mathcal{I}$, so it follows that $\otimes$ makes $\mathcal{I}$ (and $\mathcal{J}$, since the isomorphism class containing $(\mathcal{O}_S, 1)$ is a subgroup of $\mathcal{I}$) an abelian group. Also note that we have an isomorphism of pairs $(\mathcal{L}, \alpha)^{\otimes n} = (\mathcal{L}^{\otimes n}, \alpha^{\otimes n}) \xrightarrow{\alpha} (\mathcal{O}_S, 1)$, hence $\mathcal{J}$ has exponent n.

Proposition 4.2. *Let S be a scheme, and let $n \in \mathbb{N}$ be invertible on S. Then there is a canonical identification $\mathrm{H}^1_{\acute{e}t}(S, \mu_{n,S}) = \mathcal{J}$.*

Proof. Let $(\mathcal{L}, \alpha) \in \mathcal{J}$. Then take a covering $\mathcal{U} = \{U_i\}$ of S of affine open subsets, such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. Let $s_i \in \mathcal{L}(U_i)$ be a generator. Then $s_i^{\otimes n}$ is a generator of $\mathcal{L}^{\otimes n}(U_i)$, hence under α, it maps to an element f_i of $\mathcal{O}_S^\times(U_i)$. Now write $U_i = \operatorname{Spec} A_i$. Then, as we've seen before, for $B_i = A_i[T]/(T^n - f_i)$, the map $A_i \to B_i$ is a standard étale integral extension, mapping f_i to an n-th power in B_i. Hence, if we put $V_i = \operatorname{Spec} B_i$, we have an étale covering $\mathcal{V} = \{V_i \to S\}$ of S, together with trivializations $\phi_i\colon (\mathcal{L}, \alpha)|_{V_i} \to (\mathcal{O}_{V_i}, 1)$.

Now, using this covering, we can associate to $(\mathcal{L}, \alpha)$ an element of $\mathrm{H}^1_{\text{ét}}(S, \mu_{n,S})$. We do this in two ways.

Using torsors: Consider the sheaf

$$\mu_n(\mathcal{L},\alpha)\colon U \mapsto \mathrm{Iso}_U((\mathcal{O}_U,1),(\mathcal{L},\alpha)|_U)$$

on the big étale site over S. By (1), it follows that it has a free and transitive $\mu_{n,S}$-action, and since we have a trivializing cover $\mathcal{V}$ of S, it also follows that locally, $\mu_n(\mathcal{L},\alpha)$ has sections. Hence $\mu_n(\mathcal{L},\alpha)$ is a $\mu_{n,S}$-torsor, so it gives an element of $\mathrm{H}^1_{\text{ét}}(S,\mu_{n,S})$.

Using Čech cohomology: Note that we have an isomorphism

$$(\mathcal{O}_{V_i\times_S V_j},1) \xrightarrow{\mathrm{pr}_0^*\phi_i^{-1}} (\mathcal{L}|_{V_i\times_S V_j},\alpha|_{V_i\times_S V_j}) \xrightarrow{\mathrm{pr}_1^*\phi_j} (\mathcal{O}_{V_i\times_S V_j},1)$$

Using (1) it then follows that this isomorphism corresponds to an element $\zeta_{ij}\in\mu_n(V_i\times_S V_j)$. Then these ζ_{ij} give a 1-cocycle, hence gives an element of $\check{\mathrm{H}}^1(\mathcal{V},\mu_n)$. It follows that this also gives an element of $\mathrm{H}^1_{\text{ét}}(S,\mu_{n,S})$.

These two constructions give the same cohomology class. Using the Čech construction, it follows that the cohomology class of $(\mathcal{L},\alpha)\otimes(\mathcal{L}',\alpha')$ is the sum of the cohomology classes of $(\mathcal{L},\alpha)$ and $(\mathcal{L}',\alpha')$. Using the torsor construction, it follows that isomorphic pairs give rise to the same cohomology class, and also that if the cohomology class of a pair is trivial, then the pair is trivial, since torsors are trivial if and only if they have a global section. Hence we now have an injective map $\mathcal{J}\to\mathrm{H}^1_{\text{ét}}(S,\mu_{n,S})$.

So we're done if we show it is surjective. We give a sketch of this. Let $\xi\in\mathrm{H}^1_{\text{ét}}(S,\mu_{n,S})$. Then it maps to an element $\mathcal{L}\in\operatorname{Pic}S$ such that $\mathcal{L}^{\otimes n}=\mathcal{O}_S$ in $\operatorname{Pic}S$. Hence there exists a pair $(\mathcal{L},\alpha)$ with cohomology class $\xi'\in\mathrm{H}^1_{\text{ét}}(S,\mu_{n,S})$ mapping to the same element $\mathcal{L}$ of $\operatorname{Pic}S$. Now note that since ξ and ξ' map to the same element of $\operatorname{Pic}S$ (proof omitted), there is an element $f\in\Gamma(S,\mathcal{O}_S^\times)$ such that $\xi-\xi'=\delta f$. Finally, note that $(\mathcal{O}_S,f)$ has cohomology class δf (proof omitted). We deduce that ξ is the sum of two classes of pairs, hence is a class of pairs itself. □

With this, we have an explicit description of the maps

$$\Gamma(S,\mathcal{O}_S^\times)\xrightarrow{\delta}\mathrm{H}^1_{\text{ét}}(S,\mu_{n,S})\longrightarrow\operatorname{Pic}S;$$

The first map sends a map f to the class of the pair $(\mathcal{O}_S,f)$, and the second map sends the class of a pair $(\mathcal{L},\alpha)$, to the class of $\mathcal{L}$ in $\operatorname{Pic}S$.

References

[S] *The Stacks Project*, http://www.math.columbia.edu/algebraic_geometry/stacks-git/book.pdf