Blow-ups

Jinbi Jin

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1 Blow-ups

We will mostly follow the section about blow-ups in the book *Algebraic Geometry I* by U. Görtz and T. Wedhorn.

1 Introduction

Before we give the general definition, we first give a concrete example of a blow-up. Let \( X = \mathbb{A}^2_\mathbb{C} \). Then the blow-up \( \tilde{X} \) of \( X \) at the point \((0,0)\) is the reduced closed subscheme of \( \mathbb{A}^2_\mathbb{C} \times \mathbb{C} \mathbb{P}^1_\mathbb{C} \) defined by the polynomial \( y\xi - x\eta \), where \( x, y \) are coordinates of the first factor, and \( \xi, \eta \) are homogeneous coordinates of the second factor. A picture of the real points can be found for example on page 407 of *Algebraic Geometry I* by U. Görtz and T. Wedhorn, or in any other book on algebraic geometry that the reader likes.

We have a look at the some of the properties of this blow-up. Let \( \pi_i \) is the \( i \)-th projection. Then note that \( \pi_1|_{\pi_1^{-1}(\mathbb{A}^2_\mathbb{C} - \{0\})} \) defines an isomorphism with inverse \((x,y) \mapsto ((x,y), (x:y))\). Also, the exceptional divisor \( \pi_1^{-1}(0) \) is isomorphic to \( \mathbb{P}^1_\mathbb{C} \) via \( \pi_2 \) with inverse \((x:y) \mapsto ((0,0), (x:y))\). Most importantly, note that while it is impossible to define \( 0 \in \mathbb{A}^2_\mathbb{C} \) locally with a single regular element, it is possible to define \( \pi_1^{-1}(0) \) locally with a single regular element, namely with \( x\xi + y\eta \). We will later define blow-ups in the general case as being universal with respect to this property.

Now consider the reduced closed subscheme \( Y \) of \( X \) defined by the polynomial \( y^2 - x^3 - x^2 \), and its strict transform \( \tilde{Y} \), i.e. the closure of \( \pi_1^{-1}Y - \pi_1^{-1}0 \) in \( \tilde{X} \). Note that the curve \( Y \) is not regular - it has an ordinary double point at \( 0 \) - but that the curve \( \tilde{Y} \) is - it is isomorphic to \( \mathbb{A}^1_\mathbb{C} \) via \( \pi_2 \) (one other good way to think about this, is that \( \pi_2 \) gives a parametrisation of the \( (\mathbb{C}-\text{valued points of}) \) the curve \( Y \) - blow-ups resolve singularities.

2 Definition via universal property

First, we recall what an effective Cartier divisor of a scheme is.

**Definition 1.1.** Let \( X \) be a scheme. An effective Cartier divisor of \( X \) is a closed subscheme \( Z \) such that the corresponding quasi-coherent ideal \( \mathcal{I} \subseteq \mathcal{O}_X \) is locally defined by a single regular element.

Note that the empty divisor is also an effective Cartier divisor.

**Fact 1.2.** Flat pullbacks of effective Cartier divisors are effective Cartier divisors.

Now we can define the blow-up as follows.

**Definition 1.3.** Let \( X \) be a scheme, and let \( Z \) be a closed subscheme of \( X \). A blow-up of \( X \) along \( Z \) (or a blow-up of \( X \) with centre \( Z \)) is a pair \((\tilde{X}, \pi)\) - where \( \tilde{X} \) is a scheme, and \( \pi: \tilde{X} \to X \) is a...
morphism of schemes such that $\pi^{-1}Z$ is an effective Cartier divisor, satisfying the following universal property.

For any scheme $X'$ and any morphism $\pi': X' \to X$ such that $(\pi')^{-1}Z$ is an effective Cartier divisor, there exists a unique morphism $g: \tilde{X}' \to \tilde{X}$ such that $\pi' = \pi \circ g$.

If the blow-up of a scheme $X$ along a closed subscheme $Z$ exists, it is unique up to a unique isomorphism, and we denote it by $\text{Bl}_Z X$. In that case, the effective Cartier divisor $\pi^{-1}Z$ is called the exceptional divisor of $\text{Bl}_Z X$.

From the universal property, we can deduce the following.

**Proposition 1.4.** Let $X$ be a scheme, let $Z$ be a closed subscheme of $X$, and let $f: X' \to X$ be a morphism of schemes.

(i) There exists a unique morphism $\text{Bl}_Z f: \text{Bl}_{f^{-1}Z} X' \to \text{Bl}_Z X$ making the following diagram commutative.

(ii) If $f$ is flat, then diagram (1) is Cartesian.

(iii) Let $U = X - Z$. Then the restriction of $\pi$ to $\pi^{-1}U$ is an isomorphism.

**Proof.** (i) follows formally from the universal property.

For (ii), let $\tilde{X}' = \text{Bl}_Z (X) \times_X X'$, and let $p_1: \tilde{X}' \to \text{Bl}_Z (X)$, $p_2: \tilde{X}' \to X'$ be the first and second projection, respectively. Then first note that by the universal property of fibre products, there exists a unique morphism $r: \text{Bl}_{f^{-1}Z} X' \to \tilde{X}'$ such that $\text{Bl}_Z f = p_1 \circ r$, $\pi' = p_2 \circ r$.

Now $p_1$ is flat as $f$ is flat, and $E = \pi^{-1}Z$ is an effective Cartier divisor, hence $p_1^{-1}E$ is an effective Cartier divisor as well by Fact 1.2. As $\pi \circ p_1 = f \circ p_2$, it follows that $p_1^{-1}E$ is also the inverse image of $f^{-1}Z$ along $p_2$. Hence by the universal property of blow-ups of $X'$, there exists a unique morphism $r': \tilde{X}' \to \text{Bl}_{f^{-1}Z} X'$ such that $p_2 = \pi' \circ r'$. For this morphism $r'$, we also have $\text{Bl}_Z f \circ f' = p_1$, by the universal property of blow-ups of $X$. Hence diagram (2) is commutative.
As \( \pi' \circ r' \circ r = p_2 \circ r = \pi' \), we have \( r' \circ r = \text{id}_{\text{Bl}_{\pi} X} \) by the universal property of blow-ups of \( X' \). Also note that \( p_2 \circ r \circ r' = \pi' \circ r' = p_2 \), and that \( p_1 \circ r \circ r' = \text{Bl}_Z f \circ r' = p_1 \), so by the universal property of fibre products, we have \( r \circ r' = \text{id}_X \). We deduce that diagram (1) is Cartesian.

For (iii), we take for \( f \) the open immersion \( U \to X \). Note that \( \text{Bl}_{\pi^{-1} Z} U = B \otimes U = U \) (with \( \pi' = \text{id}_U \)). Hence by (ii), \( \text{Bl}_Z f : U \to \text{Bl}_Z X \) is an open immersion with image \( \pi^{-1} U \), as desired. \( \square \)

3 Construction of the blow-up

Let \( X \) be a scheme, let \( Z \) be a closed subscheme of \( X \), and let \( \mathcal{I} \subseteq \mathcal{O}_X \) be its corresponding quasi-coherent ideal. Then \( B = \bigoplus_{d \geq 0} \mathcal{I}^d \) is a graded quasi-coherent \( \mathcal{O}_X \)-algebra that is generated in degree 1. Define \( \tilde{X} = \text{Proj} B \) with structure morphism \( \pi \), where \( \text{Proj}_X \) denotes the relative projective spectrum. Moreover, let \( E = \pi^{-1} Z \).

**Proposition 1.5.** The pair \( (\tilde{X}, \pi) \) is a blow-up of \( X \) along \( Z \).

**Proof.** By Proposition 1.4(ii) applied to open immersions, it suffices to check our claim locally. So suppose \( X = \text{Spec} A \), and let \( I \) be the ideal of \( A \) such that its associated \( \mathcal{O}_X \)-module \( I \) is \( \mathcal{I} \), i.e. such that \( Z = V(I) \). Then for the graded \( A \)-module \( B = \bigoplus_{d \geq 0} \mathcal{I}^d \), its associated graded \( \mathcal{O}_X \)-module \( \tilde{B} \) is \( B \).

Let us describe \( \tilde{X} = \text{Proj} B \) locally. To this end, note that \( \tilde{X} \) is covered by \( (D(f)) \subseteq \tilde{X} \) for \( f \in I \). So let \( f \in I \). Define \( A[If^{-1}] \) to be the sub-\( A \)-algebra of \( A_f \) generated by elements of the form \( x/f \) for \( x \in I \). For a family \( (x_a)_a \) of generators for \( I \), we can describe \( A[If^{-1}] \) as \( A[[T_a]]/(fT_a - x_a) \). Then we have a canonical isomorphism \( A[If^{-1}] \to B(f) \).

Note that \( IA[If^{-1}] \) is generated as an \( A[If^{-1}] \)-module by elements \( x \in I \). Since \( x = f \cdot x/f \in fA[If^{-1}] \), it follows that \( IA[If^{-1}] \) is generated by \( f \). Moreover, \( f \) is a regular element in \( A[If^{-1}] \), as this is a sub-\( A \)-algebra of \( A_f \), in which \( f \) is invertible.

Now suppose that \( \phi : A \to C \) is a morphism of \( A \)-algebras such that \( \phi f \) is regular, and such that \( \phi f \) generates \( \phi 1 \cdot C \). Then there exists a unique \( A \)-algebra morphism \( A[If^{-1}] \to C \) compatible with \( \phi \), namely the one sending \( x/f \) to the unique element \( c \) satisfying \( \phi f \cdot c = \phi x \).

Hence for any morphism \( \tilde{X}' \to X \) making \( Z \) a Cartier divisor, by gluing the unique maps obtained above for a generating family of elements \( f \) of \( I \), we get a unique map \( \tilde{X}' \to \tilde{X} \). \( \square \)

From this description of the blow-up, we can deduce the following.

**Proposition 1.6.** Let \( X \) be a scheme, let \( Z \) be a closed subscheme of \( X \), let \( \mathcal{I} \subseteq \mathcal{O}_X \) be its corresponding quasi-coherent ideal, and let \( \pi : \text{Bl}_Z(X) \to X \) be the blow-up morphism.

(i) The exceptional divisor \( E \) is given by \( \text{Proj} \bigoplus_{d \geq 0} \mathcal{I}^d / \mathcal{I}^{d+1} \).

(ii) If \( \mathcal{I} \) is of finite type, then \( \pi \) is projective, and \( \mathcal{I} \mathcal{O}_X = \mathcal{O}_X(1) \) is very ample for \( \pi \).

(iii) If \( i : Y \to X \) is a closed immersion, then so is \( \text{Bl}_Z i : \text{Bl}_Y \otimes Z Y \to \text{Bl}_Z X \). In this situation, the closed subscheme structure corresponding to \( \text{Bl}_Y \otimes Z Y \) is called the strict transform of \( Y \) in \( \text{Bl}_Z X \).

**Proof.** For (i), note that taking the exceptional divisor is the same as taking the fibre product along the closed subscheme \( Z \), hence the same as taking the tensor product with \( \mathcal{O}_X / \mathcal{I} \). This gives the desired result.

For (ii), we check this locally. Suppose that \( X = \text{Spec} A \), and let \( I \) be the ideal of \( A \) such that its associated \( \mathcal{O}_X \)-module \( I \) is \( \mathcal{I} \), i.e. such that \( Z = V(I) \). By assumption, we may assume that \( I \) is finitely generated, say by \( f_1, \ldots, f_n \). Then \( \text{Bl}_Z X = \text{Proj} B \), where \( B = \bigoplus_{d \geq 0} \mathcal{I}^d \). As \( I \) is generated by \( f_1, \ldots, f_n \), it follows that the morphism of graded \( A \)-algebras \( A[x_1, \ldots, x_n] \to B \)
sending $x_i$ to $f_i \in B_1$ is surjective. This gives a closed immersion $\text{Bl}_Z X \to \mathbb{P}^{n-1}_A$, which makes $\mathcal{I} \mathcal{O}_Z$ a very ample invertible $\mathcal{O}_X$-module.

For (iii), note that if $Y$ is defined in $X$ by the quasi-coherent $\mathcal{O}_X$ ideal $\mathcal{J}$, then $Y \cap Z$ in $Y$ is defined by the quasi-coherent $\mathcal{O}_Y$-ideal $(\mathcal{I} + \mathcal{J})/\mathcal{J}$. We deduce that $\text{Bl}_Z i$ is given by the surjective morphism of graded $\mathcal{O}_X$-algebras $\bigoplus_{d \geq 0} \mathcal{I}^d \to \bigoplus_{d \geq 0} ((\mathcal{I} + \mathcal{J})/\mathcal{J})^d$, hence a closed immersion. □

4 Examples of blow-ups
The proof of Proposition 1.6(ii) gives a recipe to calculate the blow-up of a scheme $X$ in a closed subscheme $Z$ of which the corresponding quasi-coherent ideal is of finite type. We give some examples of this.

Example 1.7. Let us first revisit the example given in the introduction. So let $A = \mathbb{C}[x,y]$, then $X = \text{Spec} A = \mathbb{A}^2_\mathbb{C}$. We blow up $\mathbb{A}^2_\mathbb{C}$ in the point $(0,0)$ with the reduced subscheme structure, i.e. in $V(I)$ with $I = (x,y)$. Let $B = \bigoplus_{d \geq 0} I^d$, then the morphism of graded $A$-modules $A[\xi, \eta] \to B$ given by $\xi \mapsto x \in B_1$, $\eta \mapsto y \in B_1$ is surjective, with kernel generated by $y \xi - x \eta$. Hence $B \cong A[\xi, \eta]/(y \xi - x \eta)$, from which we deduce that

$$\text{Bl}_Z X = \text{Proj} A[\xi, \eta]/(y \xi - x \eta).$$

Now we look at the exceptional divisor $E$. We obtain $E$ by dividing out $B$ by the ideal $I$, and then taking Proj. Hence

$$E = \text{Proj} \mathbb{C}[\xi, \eta] \cong \mathbb{P}^1_\mathbb{C}.$$

Now let $X'$ be the reduced subscheme of $\mathbb{A}^2_\mathbb{C}$ defined by the polynomial $y^2 - x^3 - x^2$. It is singular at the point $(0,0)$. Let $A' = \mathcal{O}_X(X') = \mathbb{C}[x,y]/(y^2 - x^3 - x^2)$. We blow up $X'$ in the point $Z' = (0,0)$ with the reduced subscheme structure, i.e. $Z' = V(I')$ with $I' = (x,y)$. Let $B' = \bigoplus_{d \geq 0} (I')^d$. Then the morphism of graded $A'$-modules $A'[\xi, \eta] \to B'$ given by $\xi \mapsto x \in B'_1$, $\eta \mapsto y \in B'_1$ is surjective, with kernel generated by $y \xi - x \eta$, $y \eta - x^2 \xi - x \xi$ and $\eta^2 - x^2 \xi - \xi^2$. Hence

$$\text{Bl}_{Z'} X' = \text{Proj} A'[\xi, \eta]/(y \xi - x \eta, \xi^2 - x^2 \xi - \xi^2 \xi^2 - \xi^2) \cong \text{Spec} \mathbb{C}[\eta].$$

(The reader is invited to check on standard affine open subsets that the second generator $y \eta - x^2 \xi - x \xi$ can indeed be omitted, and to write down the isomorphism above.) Hence the strict transform of $X'$ in $\text{Bl}_Z X$ is isomorphic to $\mathbb{A}^1_\mathbb{C}$, so it is now smooth over $\mathbb{C}$. Its exceptional divisor $E'$ is $\text{Proj} \mathbb{C}[\xi, \eta]/(\xi^2 - \xi^2)$, which consists of two points, which correspond to the tangent directions of $X'$ at $(0,0)$.

Example 1.8. Let $A = \mathbb{Z}[x,y]/(y^2 + xy - x^3 - 8x^2 - 17x)$, and let $X = \text{Spec} A$. Then $X$ is regular everywhere except in the point $Z$ defined by the maximal ideal $(x,y,17)$. We blow up $X$ in $Z$. The result is

$$\text{Bl}_Z X = \text{Proj} A[\xi, \eta, \pi]/(y \xi - x \eta, 17 \xi - x \pi, 17 \eta - y \pi, \xi^2 + \xi \eta - x^2 \xi - 8 \xi^2 - \xi \pi);$$

$$E = \text{Proj} \mathbb{F}_{17}[\xi, \eta, \pi]/(\eta^2 + \xi \eta - 8 \xi^2 - \xi \pi);$$

$$\text{Bl}_{Z, \mathbb{F}_{17}} X_{\mathbb{F}_{17}} = \text{Proj} (A \otimes \mathbb{F}_{17})[\xi, \eta]/(y \xi - x \eta, \eta^2 + \xi \eta - x^2 \xi - 8 \xi^2);$$

$$E \cap \text{Bl}_{Z, \mathbb{F}_{17}} X_{\mathbb{F}_{17}} = \text{Proj} \mathbb{F}_{17}[\xi, \eta]/((\xi^2 + \xi \eta - 8 \xi^2)).$$

The exceptional divisor $E$ is a non-degenerate conic in $\mathbb{P}^2_{\mathbb{F}_{17}}$ with an $\mathbb{F}_{17}$-rational point, hence isomorphic to $\mathbb{P}^1_{\mathbb{F}_{17}}$. This exceptional divisor meets the strict transform of $X_{\mathbb{F}_{17}}$ transversally in two distinct points, and this strict transform is isomorphic to $\mathbb{A}^1_{A \otimes \mathbb{F}_{17}}$. 

□
5 Resolving singularities

A theorem by Hironaka and Abhyankar states that one can always resolve singularities on a certain class of schemes via a chain of blow-ups with smooth centres. Before we give the precise statement, we are going to define this class of schemes.

Definition 1.9. Consider the following properties of a locally Noetherian scheme $X$.

(i) For all $x \in X$ every fibre of the canonical morphism $\text{Spec} \hat{O}_{X,x} \to \text{Spec} O_{X,x}$ is geometrically regular.

(ii) For every morphism $Y \to X$ of finite type, the regular locus of $Y$ is open in $Y$.

(iii) For every morphism $Y \to X$ of finite type, and for all closed irreducible subschemes $Z \subseteq Z'$ of $Y$, all maximal chains of closed irreducible subschemes $Z = Z_0 \subset Z_1 \subset \cdots \subset Z_n = Z'$ have the same length. (In this case, $X$ is called universally catenary.)

If $X$ satisfies (i) and (ii), then $X$ is called quasi-excellent. If $X$ satisfies (i), (ii), and (iii), then $X$ is called excellent.

Most of the schemes that we typically encounter turn out to be excellent.

Fact 1.10.

(i) Suppose that $X$ is an excellent scheme, and that $Y$ is a scheme that is of finite type over $X$. Then $Y$ is also excellent.

(ii) Suppose that $R$ is a complete local Noetherian ring (e.g. a field) or a Dedekind ring of which the field of fractions has characteristic 0 (e.g. $\mathbb{Z}$), then $\text{Spec} R$ is excellent. In particular, all schemes of finite type over a ring as in (ii) are excellent.

Now we can state the theorem mentioned in the beginning of this section.

Theorem 1.11 (Hironaka (i), Abhyankar (ii)). Let $X$ be a reduced excellent scheme satisfying either of the following properties.

(i) The characteristic of $X$ is 0 (i.e. all residue fields have characteristic 0);

(ii) The dimension of $X$ is at most 2.

Then there exists a sequence of morphisms

$$X_n \xrightarrow{f_{n-1}} X_{n-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{f_0} X_0$$

such that

(a) Every $f_i$ is a blow-up with regular centre $D_i$ that is contained in the singular locus of $X_i$;

(b) Every $X_i$ is normally flat along $D_i$, i.e. the $O_{D_i}$-module $\bigoplus_{d \geq 0} T^d / T^{d+1}$ is flat;

(c) $X_n$ is regular.

In particular, the restriction of this sequence of morphisms to the inverse image of the regular locus of $X_0$ is an isomorphism.