Blow-ups

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1 Blow-ups

We will mostly follow the section about blow-ups in the book *Algebraic Geometry I* by U. Görtz and T. Wedhorn.

1 Introduction

Before we give the general definition, we first give a concrete example of a blow-up.

Let $X = \mathbb{A}^2_{\mathbb{C}}$. Then the *blow-up* $\tilde{X}$ of X at the point $(0,0)$ is the reduced closed subscheme of $\mathbb{A}^2_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ defined by the polynomial $y\xi - x\eta$, where x, y are coordinates of the first factor, and ξ, η are homogeneous coordinates of the second factor. A picture of the real points can be found for example on page 407 of *Algebraic Geometry I* by U. Görtz and T. Wedhorn, or in any other book on algebraic geometry that the reader likes.

We have a look at the some of the properties of this blow-up. Let π_i is the i-th projection. Then note that $\pi_1|_{\pi_1^{-1}(\mathbb{A}^2_{\mathbb{C}} - \{0\})}$ defines an isomorphism with inverse $(x, y) \mapsto \big((x, y), (x : y)\big)$. Also, the *exceptional divisor* $\pi_1^{-1}(0)$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ via π_2 with inverse $(x : y) \mapsto \big((0,0), (x : y)\big)$. Most importantly, note that while it is impossible to define $0 \in \mathbb{A}^2_{\mathbb{C}}$ locally with a single regular element, it is possible to define $\pi_1^{-1}(0)$ locally with a single regular element, namely with $x\xi + y\eta$. We will later define blow-ups in the general case as being universal with respect to this property.

Now consider the reduced closed subscheme Y of X defined by the polynomial $y^2 - x^3 - x^2$, and its *strict transform* $\tilde{Y}$, i.e. the closure of $\pi_1^{-1}Y - \pi_1^{-1}0$ in $\tilde{X}$. Note that the curve Y is not regular - it has an ordinary double point at 0 - but that the curve $\tilde{Y}$ is - it is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$ via π_2 (one other good way to think about this, is that π_2 gives a parametrisation of the ($\mathbb{C}$-valued points of) the curve Y) - blow-ups resolve singularities.

2 Definition via universal property

First, we recall what an effective Cartier divisor of a scheme is.

Definition 1.1. Let X be a scheme. An *effective Cartier divisor* of X is a closed subscheme Z such that the corresponding quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$ is locally defined by a single regular element.

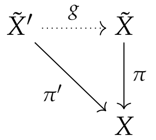
Note that the empty divisor is also an effective Cartier divisor.

Fact 1.2. *Flat pullbacks of effective Cartier divisors are effective Cartier divisors.*

Now we can define the blow-up as follows.

Definition 1.3. Let X be a scheme, and let Z be a closed subscheme of X. A *blow-up of X along Z* (or a *blow-up of X with centre Z*) is a pair $\big(\tilde{X}, \pi\big)$ - where $\tilde{X}$ is a scheme, and $\pi \colon \tilde{X} \to X$ is a morphism of schemes such that $\pi^{-1}Z$ is an effective Cartier divisor - satisfying the following universal property.

For any scheme $\tilde{X}'$ and any morphism $\pi' \colon \tilde{X}' \to X$ such that $(\pi')^{-1}Z$ is an effective Cartier divisor, there exists a unique morphism $g \colon \tilde{X}' \to \tilde{X}$ such that $\pi' = \pi \circ g$.



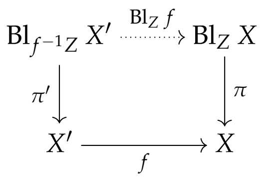
If the blow-up of a scheme X along a closed subscheme Z exists, it is unique up to a unique isomorphism, and we denote it by $\mathrm{Bl}_Z X$. In that case, the effective Cartier divisor $\pi^{-1}Z$ is called the *exceptional divisor* of $\mathrm{Bl}_Z X$.

From the universal property, we can deduce the following.

Proposition 1.4. *Let X be a scheme, let Z be a closed subscheme of X, and let $f \colon X' \to X$ be a morphism of schemes.*

(i) There exists a unique morphism $\mathrm{Bl}_Z f \colon \mathrm{Bl}_{f^{-1}Z} X' \to \mathrm{Bl}_Z X$ making the following diagram commutative.

(1)



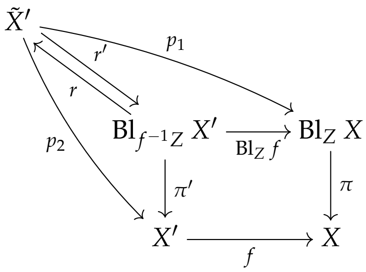
(ii) If f is flat, then diagram (1) is Cartesian.

(iii) Let $U = X - Z$. Then the restriction of π to $\pi^{-1}U$ is an isomorphism.

Proof. (i) follows formally from the universal property.

For (ii), let $\tilde{X}' = \mathrm{Bl}_Z(X) \times_X X'$, and let $p_1 \colon \tilde{X}' \to \mathrm{Bl}_Z(X)$, $p_2 \colon \tilde{X}' \to X'$ be the first and second projection, respectively. Then first note that by the universal property of fibre products, there exists a unique morphism $r \colon \mathrm{Bl}_{f^{-1}Z} X' \to \tilde{X}'$ such that $\mathrm{Bl}_Z f = p_1 \circ r$, $\pi' = p_2 \circ r$.

(2)



Now p_1 is flat as f is flat, and $E = \pi^{-1}Z$ is an effective Cartier divisor, hence $p_1^{-1}E$ is an effective Cartier divisor as well by Fact 1.2. As $\pi \circ p_1 = f \circ p_2$, it follows that $p_1^{-1}E$ is also the inverse image of $f^{-1}Z$ along p_2. Hence by the universal property of blow-ups of X', there exists a unique morphism $r' \colon \tilde{X}' \to \mathrm{Bl}_{f^{-1}Z} X'$ such that $p_2 = \pi' \circ r'$. For this morphism r', we also have $\mathrm{Bl}_Z f \circ f' = p_1$, by the universal property of blow-ups of X. Hence diagram (2) is commutative.

As $\pi' \circ r' \circ r = p_2 \circ r = \pi'$, we have $r' \circ r = \mathrm{id}_{\mathrm{Bl}_{f^{-1}Z} X'}$ by the universal property of blow-ups of X'. Also note that $p_2 \circ r \circ r' = \pi' \circ r' = p_2$, and that $p_1 \circ r \circ r' = \mathrm{Bl}_Z f \circ r' = p_1$, so by the universal property of fibre products, we have $r \circ r' = \mathrm{id}_{\tilde{X}'}$. We deduce that diagram (1) is Cartesian.

For (iii), we take for f the open immersion $U \to X$. Note that $\mathrm{Bl}_{f^{-1}Z} U = \mathrm{Bl}_{\emptyset} U = U$ (with $\pi' = \mathrm{id}_U$). Hence by (ii), $\mathrm{Bl}_Z f \colon U \to \mathrm{Bl}_Z X$ is an open immersion with image $\pi^{-1}U$, as desired. □

3 Construction of the blow-up

Let X be a scheme, let Z be a closed subscheme of X, and let $\mathcal{I} \subseteq \mathcal{O}_X$ be its corresponding quasi-coherent ideal. Then $\mathcal{B} = \bigoplus_{d \geq 0} \mathcal{I}^d$ is a graded quasi-coherent $\mathcal{O}_X$-algebra that is generated in degree 1. Define $\tilde{X} = \mathrm{Proj}_X \mathcal{B}$ with structure morphism π, where Proj_X denotes the relative projective spectrum. Moreover, let $E = \pi^{-1}Z$.

Proposition 1.5. *The pair $(\tilde{X}, \pi)$ is a blow-up of X along Z.*

Proof. By Proposition 1.4.(ii) applied to open immersions, it suffices to check our claim locally. So suppose $X = \mathrm{Spec}\, A$, and let I be the ideal of A such that its associated $\mathcal{O}_X$-module $\tilde{I}$ is $\mathcal{I}$, i.e. such that $Z = V(I)$. Then for the graded A-module $B = \bigoplus_{d \geq 0} I^d$, its associated graded $\mathcal{O}_X$-module $\tilde{B}$ is $\mathcal{B}$.

Let us describe $\tilde{X} = \mathrm{Proj}\, B$ locally. To this end, note that $\tilde{X}$ is covered by $\big(D(f)\big) \subseteq \tilde{X}$ for $f \in I$. So let $f \in I$. Define $A[If^{-1}]$ to be the sub-A-algebra of A_f generated by elements of the form x/f for $x \in I$. So for a family $(x_\alpha)_\alpha$ of generators for I, we can describe $A[If^{-1}]$ as $A[(T_\alpha)_\alpha]/\big((fT_\alpha - x_\alpha)_\alpha\big)$. Then we have a canonical isomorphism $A[If^{-1}] \to B_{(f)}$.

Note that $IA[If^{-1}]$ is generated as an $A[If^{-1}]$-module by elements $x \in I$. Since $x = f \cdot x/f \in fA[If^{-1}]$, it follows that $IA[If^{-1}]$ is generated by f. Moreover, f is a regular element in $A[If^{-1}]$, as this is a sub-A-algebra of A_f, in which f is invertible.

Now suppose that $\phi \colon A \to C$ is a morphism of A-algebras such that ϕf is regular, and such that ϕf generates $\phi I \cdot C$. Then there exists a unique A-algebra morphism $A[If^{-1}] \to C$ compatible with ϕ, namely the one sending x/f to the unique element c satisfying $\phi f \cdot c = \phi x$.

Hence for any morphism $\tilde{X}' \to X$ making Z a Cartier divisor, by gluing the unique maps obtained above for a generating family of elements f of I, we get a unique map $\tilde{X}' \to \tilde{X}$. □

From this description of the blow-up, we can deduce the following.

Proposition 1.6. *Let X be a scheme, let Z be a closed subscheme of X, let $\mathcal{I} \subseteq \mathcal{O}_X$ be its corresponding quasi-coherent ideal, and let $\pi \colon \mathrm{Bl}_Z(X) \to X$ be the blow-up morphism.*

(i) The exceptional divisor E is given by $\mathrm{Proj} \bigoplus_{d \geq 0} \mathcal{I}^d/\mathcal{I}^{d+1}$.
(ii) If $\mathcal{I}$ is of finite type, then π is projective, and $\mathcal{I}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(1)$ is very ample for π.
(iii) If $i \colon Y \to X$ is a closed immersion, then so is $\mathrm{Bl}_Z i \colon \mathrm{Bl}_{Y \cap Z} Y \to \mathrm{Bl}_Z X$. In this situation, the closed subscheme structure corresponding to $\mathrm{Bl}_{Y \cap Z} Y$ is called the strict transform *of Y in $\mathrm{Bl}_X Z$.*

Proof. For (i), note that taking the exceptional divisor is the same as taking the fibre product along the closed subscheme Z, hence the same as taking the tensor product with $\mathcal{O}_X/\mathcal{I}$. This gives the desired result.

For (ii), we check this locally. Suppose that $X = \mathrm{Spec}\, A$, and let I be the ideal of A such that its associated $\mathcal{O}_X$-module $\tilde{I}$ is $\mathcal{I}$, i.e. such that $Z = V(I)$. By assumption, we may assume that I is finitely generated, say by $f_1, \ldots, f_n$. Then $\mathrm{Bl}_Z X = \mathrm{Proj}\, B$, where $B = \bigoplus_{d \geq 0} I^d$. As I is generated by $f_1, \ldots, f_n$, it follows that the morphism of graded A-algebras $A[x_1, \ldots, x_n] \to B$

sending x_i to $f_i \in B_1$ is surjective. This gives a closed immersion $\mathrm{Bl}_Z X \to \mathbb{P}_A^{n-1}$, which makes $\mathcal{I}\mathcal{O}_{\tilde{X}}$ a very ample invertible $\mathcal{O}_{\tilde{X}}$-module.

For (iii), note that if Y is defined in X by the quasi-coherent $\mathcal{O}_X$ ideal $\mathcal{J}$, then $Y \cap Z$ in Y is defined by the quasi-coherent $\mathcal{O}_Y$-ideal $(\mathcal{I} + \mathcal{J})/\mathcal{J}$. We deduce that $\mathrm{Bl}_Z i$ is given by the surjective morphism of graded $\mathcal{O}_X$-algebras $\bigoplus_{d\geq 0} \mathcal{I}^d \to \bigoplus_{d\geq 0} \big((\mathcal{I} + \mathcal{J})/\mathcal{J}\big)^d$, hence a closed immersion. □

4 Examples of blow-ups

The proof of Proposition 1.6.(ii) gives a recipe to calculate the blow-up of a scheme X in a closed subscheme Z of which the corresponding quasi-coherent ideal is of finite type. We give some examples of this.

Example 1.7. Let us first revisit the example given in the introduction. So let $A = \mathbb{C}[x, y]$, then $X = \operatorname{Spec} A = \mathbb{A}^2_{\mathbb{C}}$. We blow up $\mathbb{A}^2_{\mathbb{C}}$ in the point $(0, 0)$ with the reduced subscheme structure, i.e. in $V(I)$ with $I = (x, y)$. Let $B = \bigoplus_{d\geq 0} I^d$, then the morphism of graded A-modules $A[\xi, \eta] \to B$ given by $\xi \mapsto x \in B_1$, $\eta \mapsto y \in B_1$ is surjective, with kernel generated by $y\xi - x\eta$. Hence $B \cong A[\xi, \eta]/(y\xi - x\eta)$, from which we deduce that

$$\mathrm{Bl}_Z X = \operatorname{Proj} A[\xi, \eta]/(y\xi - x\eta).$$

Now we look at the exceptional divisor E. We obtain E by dividing out B by the ideal I, and then taking Proj. Hence

$$E = \operatorname{Proj} \mathbb{C}[\xi, \eta] \cong \mathbb{P}^1_{\mathbb{C}}.$$

Now let X' be the reduced subscheme of $\mathbb{A}^2_{\mathbb{C}}$ defined by the polynomial $y^2 - x^3 - x^2$. It is singular at the point $(0, 0)$. Let $A' = \mathcal{O}_{X'}(X') = \mathbb{C}[x, y]/(y^2 - x^3 - x^2)$. We blow up X' in the point $Z' = (0, 0)$ with the reduced subscheme structure, i.e. $Z' = V(I')$ with $I' = (x, y)$. Let $B' = \bigoplus_{d\geq 0}(I')^d$. Then the morphism of graded A'-modules $A'[\xi, \eta] \to B'$ given by $\xi \mapsto x \in B'_1$, $\eta \mapsto y \in B'_1$ is surjective, with kernel generated by $y\xi - x\eta$, $y\eta - x^2\xi - x\xi$ and $\eta^2 - x\xi^2 - \xi^2$. Hence

$$\mathrm{Bl}_{Z'} X' = \operatorname{Proj} A'[\xi, \eta]/(y\xi - x\eta, \eta^2 - x\xi^2 - \xi^2) \cong \operatorname{Spec} \mathbb{C}[\eta].$$

(The reader is invited to check on standard affine open subsets that the second generator $y\eta - x^2\xi - x\xi$ can indeed be omitted, and to write down the isomorphism above.) Hence the strict transform of X' in $\mathrm{Bl}_Z X$ is isomorphic to $\mathbb{A}^1_{\mathbb{C}}$, so it is now smooth over $\mathbb{C}$. Its exceptional divisor E' is $\operatorname{Proj} \mathbb{C}[\xi, \eta]/(\eta^2 - \xi^2)$, which consists of two points, which correspond to the tangent directions of X' at $(0, 0)$.

Example 1.8. Let $A = \mathbb{Z}[x, y]/(y^2 + xy - x^3 - 8x^2 - 17x)$, and let $X = \operatorname{Spec} A$. Then X is regular everywhere except in the point Z defined by the maximal ideal $(x, y, 17)$. We blow up X in Z. The result is

$$\begin{aligned}
\mathrm{Bl}_Z X &= \operatorname{Proj} A[\xi, \eta, \pi]/(y\xi - x\eta, 17\xi - x\pi, 17\eta - y\pi, \eta^2 + \xi\eta - x\xi^2 - 8\xi^2 - \xi\pi);\\
E &= \operatorname{Proj} \mathbb{F}_{17}[\xi, \eta, \pi]/(\eta^2 + \xi\eta - 8\xi^2 - \xi\pi);\\
\mathrm{Bl}_{Z_{\mathbb{F}_{17}}} X_{\mathbb{F}_{17}} &= \operatorname{Proj}(A \otimes \mathbb{F}_{17})[\xi, \eta]/(y\xi - x\eta, \eta^2 + \xi\eta - x\xi^2 - 8\xi^2);\\
E \cap \mathrm{Bl}_{Z_{\mathbb{F}_{17}}} X_{\mathbb{F}_{17}} &= \operatorname{Proj} \mathbb{F}_{17}[\xi, \eta]/(\eta^2 + \xi\eta - 8\xi^2) = \operatorname{Proj} \mathbb{F}_{17}[\xi, \eta]/\big((\xi - 6\eta)(\xi + 7\eta)\big).
\end{aligned}$$

The exceptional divisor E is a non-degenerate conic in $\mathbb{P}^2_{\mathbb{F}_{17}}$ with an $\mathbb{F}_{17}$-rational point, hence isomorphic to $\mathbb{P}^1_{\mathbb{F}_{17}}$. This exceptional divisor meets the strict transform of $X_{\mathbb{F}_{17}}$ transversally in two distinct points, and this strict transform is isomorphic to $\mathbb{A}^1_{A\otimes\mathbb{F}_{17}}$.

5 Resolving singularities

A theorem by Hironaka and Abhyankar states that one can always resolve singularities on a certain class of schemes via a chain of blow-ups with smooth centres. Before we give the precise statement, we are going to define this class of schemes.

Definition 1.9. Consider the following properties of a locally Noetherian scheme X.

(i) For all $x \in X$ every fibre of the canonical morphism $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to \operatorname{Spec} \mathcal{O}_{X,x}$ is geometrically regular.
(ii) For every morphism $Y \to X$ of finite type, the regular locus of Y is open in Y.
(iii) For every morphism $Y \to X$ of finite type, and for all closed irreducible subschemes $Z \subseteq Z'$ of Y, all maximal chains of closed irreducible subschemes $Z = Z_0 \subset Z_1 \subset \cdots \subset Z_n = Z'$ have the same length. (In this case, X is called *universally catenary*.)

If X satisfies (i) and (ii), then X is called *quasi-excellent*. If X satisfies (i), (ii), and (iii), then X is called *excellent*.

Most of the schemes that we typically encounter turn out to be excellent.

Fact 1.10.

(i) Suppose that X is an excellent scheme, and that Y is a scheme that is of finite type over X. Then Y is also excellent.
(ii) Suppose that R is a complete local Noetherian ring (e.g. a field) or a Dedekind ring of which the field of fractions has characteristic 0 (e.g. $\mathbb{Z}$), then $\operatorname{Spec} R$ is excellent.

In particular, all schemes of finite type over a ring as in (ii) are excellent.

Now we can state the theorem mentioned in the beginning of this section.

Theorem 1.11 (Hironaka (i), Abhyankar (ii))**.** *Let X be a reduced excellent scheme satisfying either of the following properties.*

(i) The characteristic of X is 0 (i.e. all residue fields have characteristic 0);
(ii) The dimension of X is at most 2.

Then there exists a sequence of morphisms

$$X_n \xrightarrow{f_{n-1}} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{f_0} X_0$$

such that

(a) Every f_i is a blow-up with regular centre D_i that is contained in the singular locus of X_i;
(b) Every X_i is normally flat *along D_i, i.e. the $\mathcal{O}_{D_i}$-module $\bigoplus_{d \geq 0} \mathcal{I}^d / \mathcal{I}^{d+1}$ is flat;*
(c) X_n is regular.

In particular, the restriction of this sequence of morphisms to the inverse image of the regular locus of X_0 is an isomorphism.