Introduction

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1 Examples of fundamental groups

In this first section, I will give some examples of fundamental groups that the reader may already know.

What is a "fundamental group", say of an object X? In a way, looking at the case of the fundamental group of topological spaces, it should be a group "classifying" in some sense all "covers" of X. This is a statement that will be made more precise later on.

In a lot of cases, this fundamental group will be a so-called *profinite group*, so we start by reviewing them.

1.1 Review: Profinite groups

Definition 1. A *topological group* is a quadruple $(G, \mu, \iota, 1)$ with G a topological space, $\mu\colon G \times G \to G$ and $\iota\colon G \to G$ continuous maps, and $1 \in G$, satisfying the usual axioms for a group. A *morphism* of topological groups is a continuous morphism of groups.

Definition 2. Let G be a topological group. A *continuous G-set* is a pair (X, a) with X a discrete set, and $a\colon G \times X \to X$ a continuous map defining an action of G on X.

Definition 3. Let $\mathcal{I}$ be a small category, and let $\mathcal{C}$ be a category. A *diagram* with *index category* $\mathcal{I}$ is a functor $D\colon \mathcal{I} \to \mathcal{C}$. It is *filtered* if it satisfies the following three properties.

(a) $\mathcal{I}$ has at least one object.
(b) For any two objects x, y of $\mathcal{I}$, there exists an object z of $\mathcal{I}$ and two morphisms $x \to z$, $y \to z$.
(c) For any two objects x, y of $\mathcal{I}$ and any two morphisms $a, b\colon x \to y$, there exists an object z of $\mathcal{I}$ and a morphism $c\colon y \to z$ such that $D(ca) = D(cb)$.

It is *cofiltered* if its dual $D^{\text{op}}\colon \mathcal{I}^{\text{op}} \to \mathcal{C}^{\text{op}}$ is filtered.

The index category $\mathcal{I}$ is *filtered*, resp. *cofiltered* if the identity functor $\mathcal{I} \to \mathcal{I}$ is.

We note that if the index category of a diagram is filtered, resp. cofiltered, then so is the diagram.

Example 4. Let I be a partially ordered set, and let $\mathcal{I}$ be the category whose objects are the elements of $\mathcal{I}$, and where there exists a unique morphism $x \to y$ if and only if $x \leq y$. Then I is (co)directed if and only if $\mathcal{I}$ is (co)filtered.

Definition 5. A *profinite group* is a cofiltered limit (i.e. a limit of a cofiltered diagram) of discrete finite groups in the category of topological groups.

Cofiltered limits are the same as limits of diagrams arising from codirected sets ("projective" or "inverse systems") (see Stacks 0032), so one can also characterise profinite groups as *projective limits* (i.e. limits of projective systems) of discrete finite groups, which give profinite groups their name.

Example 6. Let G be a group. Let $I = \{N : N \triangleleft G,\ [G : N] \text{ finite}\}$, partially ordered by inclusion, and note that it is codirected. Consider the cofiltered diagram D from I to the category of topological groups defined by $D(N) = G/N$ with the discrete topology, and by setting $D(N \to N')$ equal to the natural map $G/N \to G/N'$. The limit of the diagram D is the *profinite completion* $\hat{G}$ of G, and there is a natural morphism of abstract groups $G \to \hat{G}$.

Fact 7. *A topological group is profinite if and only if it is Hausdorff, compact, and totally disconnected.*

1.2 Infinite Galois theory

Our first example is Galois theory. The main objects of study here are the finite separable field extensions. We recall:

Definition 8. Let k be a field, let l be a Galois (i.e. normal, separable) extension of k. Then the *Galois group* $\mathrm{Gal}(l/k)$ of the extension is $\mathrm{Aut}_k(l)$.

We give it the structure of a profinite group as follows. Let $I = \{e : e \subseteq l, e \text{ finite Galois over } k\}$, ordered by inclusion. It is directed, so the corresponding opposite category is codirected. Consider the cofiltered diagram D from I^{op} to the category of topological groups defined by $D(e) = \mathrm{Gal}(e/k)$, and by setting $D(e' \to e)$ equal to the restriction map $\mathrm{Gal}(e'/k) \to \mathrm{Gal}(e/k)$.

Theorem 9. *As abstract groups, we have* $\mathrm{Gal}(l/k) = \lim_{e \in I^{\mathrm{op}}} \mathrm{Gal}(e/k)$.

So we give $\mathrm{Gal}(l/k)$ the structure of a profinite group via this isomorphism. As in the finite case, we have a main theorem of Galois theory.

Theorem 10. *Let k be a field, let l be a Galois extension of k with Galois group G. Then the map*

$$\{e : e \subseteq l,\ e \text{ extension of } k\} \to \{H : H < G \text{ closed}\}$$
$$e \mapsto \mathrm{Gal}(l/e)$$

has inverse $H \mapsto l^H$. Moreover, finite extensions (of degree d) correspond to open subgroups (of index d), and Galois extensions correspond to normal subgroups.

Now let k be a field, and fix a separable closure k^{sep} of k. Denote by G_k the absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ over k. Then G_k is the *fundamental group* of k, in the following sense.

Theorem 11. *Let k be a field. Let $\mathcal{E}$ be the category of finite separable extensions of k, and let $\mathcal{S}$ be the category of finite continuous transitive G_k-sets. Then the functor $\mathcal{E}^{\mathrm{op}} \to \mathcal{S}$, $e \mapsto \mathrm{Hom}_k(e, k^{\mathrm{sep}})$ is an equivalence of categories, with quasi-inverse $X \mapsto \mathrm{Hom}_{G_k}(X, k^{\mathrm{sep}})$.*

1.3 Topological covers

Our second example is that of the fundamental group of a pointed topological space. One approach to defining this group, which works in general, is to take homotopy classes of loops, but this approach is not interesting in the context of this seminar. Another approach, which is more interesting to us, is the define the group via covers, but we need to restrict to "nicer" topological spaces in that case. We recall:

Definition 12. Let X be a topological space. A *covering map*, or *cover* for short, is a morphism $p\colon Y \to X$ of topological spaces such that for all $x \in X$ there exists an open neighbourhood U of x such that $p^{-1}(U)$ is the disjoint union of open sets V_i such that $p|_{V_i}$ is a homeomorphism to U for all i. If X is connected, then the cover p is *universal* if Y is connected and simply connected.

A universal cover of a connected topological space X satisfies the following property; for all covers $\varphi\colon Z \to X$, there exists a cover $\overline{p}\colon Y \to Z$ such that $p = \varphi\overline{p}$.

Definition 13. Let $p\colon Y \to X$ be a cover. A *deck transformation* is an element of $\mathrm{Aut}_X(Y)$. If p is universal, then the group of deck transformations is the *fundamental group* of X, and we denote it by $\pi(X)$.

It classifies the covers of X admitting a universal cover as follows.

Theorem 14. *Let X be a connected topological space admitting a universal cover, and let $x \in X$. Let $\mathcal{C}$ denote the category of covers of X. Let $\mathcal{S}$ denote the category of $\pi(X)$-sets. Then the functor $\mathcal{C} \to \mathcal{S}$, $(p\colon Y \to X) \mapsto p^{-1}(x)$ is an equivalence of categories.*

The following generalisation is much less known, but can be proved with the methods of this seminar.

Theorem 15. *Let X be a connected topological space. Let $\mathcal{C}$ denote the category of finite covers of X. Then there exists a profinite group π (unique up to isomorphism) such that there exists an equivalence of categories $\mathcal{C} \to \mathcal{S}$. Here, $\mathcal{S}$ is the category of finite continuous π-sets. Moreover, if X admits a universal cover, then $\pi \cong \hat{\pi}(X)$.*

2 Main theorems, outline of seminar

This seminar's first main theorem will be a simultaneous generalisation of our both examples. The objects of study are connected schemes and "covers" of them. Though, the direct scheme-theoretic analogue of the notion of a cover of topological spaces is not a good one. The following example should illustrate this.

Example 16. Let $\mathbb{G}_{m,\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, x^{-1}]$, and the endomorphism φ given on rings as $\mathbb{C}[x, x^{-1}] \to \mathbb{C}[x, x^{-1}]$, $x \mapsto x^2$. The induced map $\mathbb{G}_{m,\mathbb{C}}(\mathbb{C}) \to \mathbb{G}_{m,\mathbb{C}}(\mathbb{C})$ is a cover under the complex topology, but not under the Zariski topology; $\mathbb{G}_{m,\mathbb{C}}(\mathbb{C})$ with the Zariski topology is irreducible, so any non-empty open subset is connected. But then on any non-empty open $U \subseteq \mathbb{G}_{m,\mathbb{C}}(\mathbb{C})$, the map $\varphi^{-1}(U) \to U$ is non-injective, so φ is not a cover for the Zariski topology.

The correct generalisation of a cover turns out to be the notion of a "finite étale" morphism of schemes.

Definition 17. A morphism $f\colon X \to S$ of schemes is *finite* if for all $s \in S$ and all affine open neighbourhoods $V \ni s$, $U = f^{-1}(V)$ is affine, and the morphism $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ is finite.

We first give a definition of an étale morphism purely in terms of the functor of points.

Definition 18. A morphism $f\colon X \to S$ of schemes is *formally étale* if for all rings A, all morphisms $\operatorname{Spec} A \to S$, and all ideals $I \subseteq A$ with $I^2 = 0$, the map $X_S(A) \to X_S(A/I)$ is a bijection.

Definition 19. A morphism $f\colon X \to S$ of schemes is *locally of finite presentation* if it *commutes with cofiltered limits*, i.e. if for all cofiltered diagrams D (with index category $\mathcal{I}$) of S-schemes that are affine schemes, we have

$$X\big(\lim_i D(i)\big) = \operatorname{colim}_i X\big(D(i)\big).$$

Remark 20. Note the difference between "S-schemes that are affine schemes" and "affine S-schemes".

Definition 21. A morphism $f\colon X \to S$ of schemes is *étale* if it is locally of finite presentation and formally étale.

We can characterise this notion in a more concrete way.

Proposition 22. *A morphism $f\colon X \to S$ of schemes is étale if and only if for all $x \in X$, there exist affine open neighbourhoods $U \ni x$ and $V \ni f(x)$ with $f(U) \subseteq V$ such that $\mathcal{O}_X(U)$ is isomorphic to an $\mathcal{O}_S(V)$-algebra of the form*

$$A = \mathcal{O}_S(V)[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$$

with $\det(\partial f_j/\partial x_i)_{i,j=1}^n \in A^\times$.

Example 23. Returning to Example 16, we show that the morphism φ given there is a finite (and) étale morphism of schemes. For this, we note that the given $\mathbb{C}[x, x^{-1}]$-algebra structure is isomorphic to $A = \mathbb{C}[x, x^{-1}, t]/(t^2 - x)$, and $\partial(t^2 - x)/\partial t = 2t \in A^\times$. Moreover, A has $\mathbb{C}[x, x^{-1}]$-basis $1, t$. Hence the morphism φ of Example 16 is indeed finite étale.

In the following, we will also need the notion of degree.

Definition 24. Let $f\colon X \to S$ be a finite étale morphism, and let $s \in S$. The *degree* $(\deg f)(s)$ of f at s is $\dim_{\kappa(s)} \mathcal{O}_{X_s}(X_s)$. Here, $X_s = X \times_S \operatorname{Spec} \kappa(s)$.

The degree of a finite étale morphism $f\colon X \to S$ turns out to be a *continuous map* $S \to \mathbb{Z}$. In particular, if S is connected, then $\deg f$ is constant.

We can now formulate the first main theorem of the seminar.

Theorem 25. *Let S be a connected scheme. Let $\mathcal{E}$ be the category of finite étale S-schemes. Then there exists a profinite group $\pi(S)$, unique up to isomorphism, such that there exists an equivalence of categories $F\colon \mathcal{E} \to \mathcal{S}$, where $\mathcal{S}$ is the category of finite continuous $\pi(S)$-sets. Moreover, if X is a finite étale S-scheme, then* $\deg X = \#F(X)$.

This theorem has the following immediate and useful corollary.

Corollary 26. *Let S be a connected scheme, and let X and Y be finite étale S-schemes. Then* $\operatorname{Hom}_S(X, Y)$ *is finite.*

Proof. By the equivalence of categories, we have

$$\operatorname{Hom}_S(X, Y) = \operatorname{Hom}_{\pi(S)}\big(F(X), F(Y)\big) \subseteq \operatorname{Hom}\big(F(X), F(Y)\big),$$

which clearly is finite. ☐

Strategy of proof of Theorem 25. The strategy is to abstract the notion of a cover into a so-called *Galois category*, and prove Theorem 25 for all Galois categories. After that, we show that the category of finite étale schemes over a fixed connected scheme is a Galois category, finishing the proof. ☐

The second main theorem of the seminar, is about a finiteness result for the profinite group π of Theorem 25. Of course, an infinite profinite group π is uncountable, so it cannot be finitely generated as an abstract group. We do have the following notion.

Definition 27. A topological group G is *topologically finitely generated* if there exists a dense finitely generated subgroup of G.

Theorem 28. *Let k be an algebraically closed field. Let S be a connected, proper k-scheme. Then the $\pi(S)$ is topologically finitely generated.*

Similarly as before, we have the following immediate corollary.

Corollary 29. *Let k be an algebraically closed field. Let S be a connected, proper k-scheme, and let d be a positive integer. Then the set of isomorphism classes of finite étale S-schemes of degree d is finite.*

Proof. By the equivalence of categories, it suffices to show that there are only finitely many continuous $\pi(S)$-actions on a set X with $\#X = d$. Let $F \subseteq \pi(S)$ be a finite set of topological generators, and let A be the set of continuous $\pi(S)$-actions on X. Suppose $a, b \in A$ are two actions that coincide on $F \times X$. Then they also coincide on $\langle F \rangle \times X$. As $\pi(S) \times X$ and X are Hausdorff, it follows that $a = b$. We deduce that the natural map $A \to \mathrm{Aut}(X)^F$ is injective, and the latter set is clearly finite. □

Strategy of proof of Theorem 28.

Step 1: Prove Theorem 28 for S a smooth projective curve over k with characteristic 0.

This is done by using the *Riemann Existence Theorem* to "identify" the étale fundamental group of S with the (profinite) topological fundamental group of the "analytification" of S, which we know explicitly.

Step 2: Prove Theorem 28 for S a smooth projective curve over k with positive characteristic.

This is done by defining a smooth projective curve $\mathcal{S}$ over the ring of Witt vectors W_k of k such that $\mathcal{S} \times_{W_k} \operatorname{Spec} k = S$, and then showing that there exists a surjective morphism $\pi(\mathcal{S}_K) \to \pi(S)$, where K is an algebraic closure of the fraction field of W_k. (Keyword: *specialisation theory*)

Step 3: Prove Theorem 28 for S a normal projective k-scheme.

This is done by induction on the dimension. Suppose that $\dim S \geq 2$, and that we have a closed immersion $S \to \mathbb{P}^r_k$. For a hyperplane H of $\mathbb{P}^r_k$ not containing S, take $X = S \times_{\mathbb{P}^r_k} H$. By a variant of *Bertini's Theorem*, for some H, X will again be normal and projective. By the induction hypothesis, it then suffices to show that there exists a surjective morphism $\pi(X) \to \pi(S)$.

Step 4: Reduce remaining cases to the normal projective case.

By *Chow's Lemma*, there exists a normal and projective k-scheme X, together with a surjective morphism $X \to S$. We then use *descent* techniques to relate $\pi(S)$ to the étale fundamental groups of the connected components of X.

□