Polarisations of abelian varieties

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Last compile: 4^{th} December, 2013; 15:44

We fix a field k. For those who know schemes: a variety is a geometrically integral, separated k-scheme of finite type. For those who do not know schemes: we assume that varieties are geometrically irreducible.

0 Introduction: the moduli space of elliptic curves with 3-torsion

This is Section 2.2.10 of [K-Ma].

Let k be a field of characteristic not equal to 3. We will construct a *moduli space* of pairs (E, α), where $\alpha\colon (\mathbb{Z}/3\mathbb{Z})^2 \to E(k)[3]$ is an isomorphism of groups, or equivalently, of triples (E, P, Q) with $P, Q \in E(k)[3]$ such that $P \neq 0$ and $Q \notin \{0, P, 2P\}$.

So let (E, P, Q) be as above, and suppose that E is given by the (affine) Weierstrass equation

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6.$$

As the characteristic of k is not 3, we can put $X' = X + \frac{1}{3}a_2$ to get an equation of the form

$$Y^2 + a_1'X'Y + a_3'Y = X'^3 + a_4'X' + a_6'.$$

As $3P = 0$, there exists a rational function with divisor $3\langle P\rangle - 3\langle 0\rangle$, which is therefore a linear combination of $1, X', Y$. There is a unique such function of the form $Y + aX' + b$. Taking this to be Y', we get an equation of the form

$$Y'^2 + a_1''X'Y' + a_3''Y' = X'^3.$$

Note that P is now equal to $(0, 0)$, by our choice of Y'.

Now we consider the point Q. Let $Y' - AX' - B$ be the unique function having divisor $3\langle Q\rangle - 3\langle 0\rangle$. Suppose for a contradiction that $A = 0$. Then $Y' - B$ has a triple zero at Q, so by substituting $Y' = B$ in the equation, we see that $X'^3 - a_1''BX' - B^2 - a_3''B$ has a triple zero. Hence the X'-coordinate of Q is 0, which implies that $Q \in \{0, P, 2P\}$, contradicting our assumption. Hence $A \neq 0$, so taking $x = A^{-2}X'$ and $y = A^{-3}Y'$, we get an equation of the form

$$y^2 + b_1xy + b_3y = x^3.$$

Moreover, as E is smooth, we know that $(b_1^3 - 27b_3)b_3 \neq 0$.

It then follows that there is a unique b such that $y - x - b$ has divisor $3\langle Q\rangle - 3\langle 0\rangle$. This implies that the polynomial

$$x^3 - (1 + b_1)x^2 - (2b + bb_1 + b_3)x - (b^2 + bb_3)$$

has a triple zero, say at c. It follows that we have the following system of equations.

$$\begin{aligned} 3c &= 1 + b_1 \\ -3c^2 &= 2b + bb_1 + b_3 \\ c^3 &= b^2 + bb_3 \end{aligned}$$

Hence $b_1 = 3c - 1$ and $b_3 = -3c^2 - 3bc - b$, so the bottom equation becomes $(b+c)^3 = b^3$.

Now let $Y^{\mathrm{n}}(3)$ be the variety over k with coordinate ring

$$A = k\big[b, c, c^{-1}, (3c^2 + 3bc + b)^{-1}, (27c^3 + 54c^2 + 81bc + 9c + 27b - 1)^{-1}\big] / \big((b+c)^3 - b^3\big).$$

Any point (s,t) on $Y^{\mathrm{n}}(3)$ corresponds to the elliptic curve $E_{s,t}$ over k defined by the affine Weierstrass equation

$$Y^2 + (3t-1)XY - (3t^2 + 3st + s)Y = X^3,$$

together with the two points $P_{s,t} = (0,0)$, $Q_{s,t} = (t, s+t)$. Moreover, for every triple (E, P, Q), there exist unique s, t such that there exists an isomorphism $(E, P, Q) \to (E_{s,t}, P_{s,t}, Q_{s,t})$ (which is unique). Because of this, we call $Y^{\mathrm{n}}(3)$ the *moduli space of elliptic curves with a naive level* 3 *structure*.

We do want a theory of moduli spaces for more general abelian varieties as well, but rigidity (the lack of non-trivial automorphisms) is essential. In general, abelian varieties have too many automorphisms for this to happen, even if we fix the n-torsion.

Example 1. Let E be an elliptic curve over a field k, and consider the abelian variety $E \times E$. Then for all multiples m of $n \in \mathbb{Z}_{>0}$, we have the automorphism $(x, y) \mapsto (x + my, y)$ that fixes the n-torsion of E. So we have infinitely many automorphisms of $E \times E$ fixing the n-torsion.

In this talk, we define extra structures (so-called *polarisations*) on abelian varieties that solve this problem, enable us to develop a theory of moduli spaces for general abelian varieties as well.

0.1 Preliminaries on line bundles

Let $(X, \mathcal{O}_X)$ be a ringed space. For example, X is an (algebraic) variety over k and $\mathcal{O}_X$ is the sheaf of regular functions on it; or X is a complex manifold and $\mathcal{O}_X$ is the sheaf of holomorphic functions on X.

Definition 2. An *$\mathcal{O}_X$-module* is a triple $(\mathcal{F}, +, \cdot)$ of a sheaf $\mathcal{F}$ of abelian groups on X, and two morphisms $+\colon \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ and $\cdot\colon \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$ such that for all $U \subseteq X$, $\big(\mathcal{F}(U), +(U), \cdot(U)\big)$ is an $\mathcal{O}_X(U)$-module.

A *morphism* $\mathcal{F} \to \mathcal{F}'$ of $\mathcal{O}_X$-modules is a morphism of sheaves of abelian groups compatible with the addition and scalar multiplication morphisms.

An *invertible $\mathcal{O}_X$-module* (also *invertible sheaf on* X or *line bundle on* X) is an $\mathcal{O}_X$-module $\mathcal{L}$ such that there exists an open cover $\mathcal{U}$ of X such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$ as $\mathcal{O}_X|_U$-modules, for all $U \in \mathcal{U}$.

The *Picard group* $\operatorname{Pic} X$ of X is the set of isomorphism classes of invertible sheaves on X.

Example 3. Suppose that X is smooth variety over k, and let D be a (Weil) divisor on X. Then we can attach to all $U \subseteq X$ open, the $\mathcal{O}_X(U)$-module

$$\mathcal{L}(D)(U) = \big\{ f \in k(X)^\times : \operatorname{div} f + D \geq 0 \big\} \cup \{0\}.$$

Then $\mathcal{L}(D)$ is a line bundle on X, the *line bundle associated with* D.

Moreover, the map $\operatorname{Cl} X \to \operatorname{Pic} X$, $D \mapsto \mathcal{L}(D)$ is an isomorphism of groups.

We now give the group structure on $\operatorname{Pic} X$. Let $\mathcal{L}$ and $\mathcal{L}'$ be two line bundles on X. Then their *tensor product* $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is the sheaf associated to the presheaf $U \mapsto \mathcal{L}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}'(U)$. It is again a line bundle, and $\operatorname{Pic} X$ is a group under taking tensor products.

We next define pullbacks of line bundles along a morphism. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. For example, f is a morphism of (algebraic) varieties over a field k, or f is a morphism of complex manifolds.

We first give the definition of a morphism between sheaves of modules on different topological spaces. The virtue of this definition is that compositions are more easily defined.

Definition 4. Let $\mathcal{F}$ and $\mathcal{G}$ be an $\mathcal{O}_X$-module and an $\mathcal{O}_Y$-module, respectively. An *f-morphism* φ from $\mathcal{G}$ to $\mathcal{F}$ is a collection of, for all $U \subseteq X$ and $V \subseteq Y$ open such that $f(U) \subseteq V$, morphisms $\varphi_U^V \colon \mathcal{G}(V) \to \mathcal{F}(U)$ with the following properties.

- For all $U \subseteq X$ and $V \subseteq Y$ open such that $f(U) \subseteq V$, and all $g \in \mathcal{O}_X(U)$ and $x \in \mathcal{G}(V)$, we have $\varphi_U^V(gx) = f^\#(g)|_U \varphi_U^V(x)$.
- For all $U' \subseteq U \subseteq X$ and $V' \subseteq V \subseteq Y$ open such that $f(U) \subseteq V$ and $f(U') \subseteq V'$, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{G}(V) & \xrightarrow{\varphi_U^V} & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{G}(V') & \xrightarrow{\varphi_{U'}^{V'}} & \mathcal{F}(U') \end{array}$$

This allows us to define pullbacks of $\mathcal{O}_X$-modules via a universal property.

Definition 5. Let $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then a *pullback* of $\mathcal{G}$ along f is a pair $(f^*\mathcal{G}, f^*)$ of a line bundle $f^*\mathcal{G}$ on X and an f-morphism $f^* \colon \mathcal{G} \to f^*\mathcal{G}$ such that for all $\mathcal{O}_X$-modules $\mathcal{F}$, and f-morphisms $s \colon \mathcal{G} \to \mathcal{F}$, there exists a unique (id_X-)morphism $\bar{s} \colon f^*\mathcal{G} \to \mathcal{F}$ such that $s = \bar{s}f^*$.

So if pullbacks exist, they are unique up to a unique isomorphism.

Fact 6. *Let $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then for all $\mathcal{O}_Y$-modules $\mathcal{G}$, its pullback $f^*\mathcal{G}$ exists, and f^* induces a morphism of groups* $\operatorname{Pic} Y \to \operatorname{Pic} X$.

We now define the notion of ample and very ample line bundles on varieties.

Definition 7. Let X be a variety over k. A line bundle $\mathcal{L}$ on X is *very ample* if there exists a closed immersion (i.e. a closed map that is an isomorphism onto its image) $f \colon X \to \mathbb{P}_k^n$ for some $n \in \mathbb{Z}$ such that $\mathcal{L} \cong f^*\mathcal{L}(H)$ with H a hyperplane in $\mathbb{P}_k^n$.

A line bundle $\mathcal{L}$ on X is *ample* if there exists a positive integer n such that $\mathcal{L}^{\otimes n}$ is very ample.

0.2 Preliminaries on abelian varieties

Definition 8. An *abelian variety* over k is a proper smooth group variety over k.

Lemma 9 (Rigidity Lemma)**.** *Let X be a proper variety over k, and let Y and Z be any variety over k. Let $f \colon X \times Y \to Z$ be a morphism such that there exist $y_0 \in Y$, $z_0 \in Z$ with the property that $f(X \times \{y_0\}) = \{z_0\}$. Then there is a morphism $g \colon Y \to Z$ such that $f = g \circ \pi_2$ (where $\pi_2 \colon X \times Y \to Y$ is the projection).*

Proof. Let $x_0 \in X$, and set $g \colon Y \to Z$, $y \mapsto f(x_0, y)$. We prove that $f = g \circ \pi_2$ by proving for a non-empty open subset V of Y that $f|_{X \times V} = (g \circ \pi_2)|_{X \times V}$.

Let U be an affine open neighbourhood of z_0 in Z, let $F = Z - U$, and let $G = \pi_2(f^{-1}(F))$. As X is proper, G is a closed subset of Y. Note that $y_0 \notin G$ as $f(X \times \{y_0\}) = \{z_0\} \in U$. Hence $V = Y - G$ is a non-empty open subset of Y.

Now note that for all $y \in V$, the set $X \times \{y\}$ is mapped by f into U. As X is proper, it follows that $f|_{X \times \{y\}}$ is constant. We deduce that for all $x \in X$ and $y \in V$, we have

$$f(x, y) = f(x_0, y) = g \circ \pi_2(x, y),$$

as desired. $\square$

Corollary 10. *Let A and B be abelian varieties over k, and let $f \colon A \to B$ be a morphism of varieties such that $f(0) = 0$. Then f is a morphism of abelian varieties.*

Proof. Consider the morphism $\delta\colon A \times A \to B$, $(a, a') \mapsto f(a+a') - \big(f(a) + f(a')\big)$, and note that $\varphi(A \times \{0\}) = \{0\}$, hence we can apply the Rigidity Lemma on δ. Moreover, $\varphi(\{0\} \times A) = \{0\}$, so δ factors via the zero map $A \to B$, and hence itself is the zero map, as desired. □

Corollary 11. *Abelian varieties are commutative group varieties.*

Proof. This is simply the fact that the inverse map on an abelian variety is a morphism of groups by the above. □

On abelian varieties, the notion of an ample line bundle has the following characterisation.

Theorem 12. *Let A be an abelian variety over k, and let $\mathcal{L}$ be a line bundle on A. Then $\mathcal{L}$ is ample if and only if $\{x \in A : \tau_x^*\mathcal{L} = \mathcal{L}\}$ is finite, where τ_x is the translation-by-x map on A.*

1 Polarisations, algebraically

References: [vdG-Mo], [O]

1.1 Relative Picard groups

We first look at Picard groups in a "relative" setting.

Write, for varieties X and T over k, π_T for the second projection $X \times T \to T$.

Definition 13. Let X and T be varieties over k. The *relative Picard group* $\operatorname{Pic}(X \times T/T)$ is the quotient group $\operatorname{Pic}(X \times T)/\pi_T^* \operatorname{Pic}(T)$.

This construction is functorial. If $t\colon T \to U$ is a morphism of varieties, then we have the following commutative diagram.

$$\begin{array}{ccc} \operatorname{Pic}(X \times U) & \xrightarrow{(\operatorname{id} \times t)^*} & \operatorname{Pic}(X \times T) \\ \uparrow{\scriptstyle \pi_U^*} & & \uparrow{\scriptstyle \pi_T^*} \\ \operatorname{Pic} U & \xrightarrow{\quad t^* \quad} & \operatorname{Pic} T \end{array}$$

It induces a morphism $\operatorname{Pic}(X \times U/U) \to \operatorname{Pic}(X \times T/T)$ also denoted by $(\operatorname{id} \times t)^*$.

Definition 14. Let X and T be varieties over k. Let $P \in \operatorname{Pic}(X \times T/T)$. Then P is *algebraically equivalent to* 0 if there exist a variety T' over k, two morphisms $t_1, t_2\colon T \to T'$, and $Q \in \operatorname{Pic}(X \times T'/T')$ such that $(\operatorname{id} \times t_1)^*Q = P$ and $(\operatorname{id} \times t_2)^*Q = 0$ in $\operatorname{Pic}(X \times T/T)$.

The set of $P \in \operatorname{Pic}(X \times T/T)$ algebraically equivalent to 0 is denoted $\operatorname{Pic}^0(X \times T/T)$.

We give some elementary properties that are best left as exercise to the reader.

Proposition 15. *Let X be a variety over k. Let $t\colon T \to U$ be a morphism of varieties.*

- *The subset* $\operatorname{Pic}^0(X \times T/T)$ *is a subgroup of* $\operatorname{Pic}(X \times T/T)$.
- *The morphism* $(\operatorname{id} \times t)^*\colon \operatorname{Pic}(X \times U/U) \to \operatorname{Pic}(X \times T/T)$ *of groups induces a morphism* $\operatorname{Pic}^0(X \times U/U) \to \operatorname{Pic}^0(X \times T/T)$ *of groups.*

Remark 16. Usually, a theory of a relative Picard group is developed using the concept of a *rigidified line bundle* instead. But this is equivalent to the one presented here, as one can associate with each $P \in \operatorname{Pic}(X \times T/T)$ an, up to a unique isomorphism, unique line bundle (via a rigidification map $\mathcal{L} \mapsto \mathcal{L} \otimes \pi_T^* 0^* \mathcal{L}^{-1}$).

1.2 The dual abelian variety

We define the dual abelian variety of an abelian variety A over k by a universal property.

Definition 17. Let A be an abelian variety over k. A *dual abelian variety* of A is a pair $(A^t, \mathcal{P})$ of an abelian variety A^t, and an element $\mathcal{P} \in \mathrm{Pic}^0(A \times A^t / A^t)$ (called the *Poincaré class*) with the following properties.

- For all varieties T over k and all $P \in \mathrm{Pic}^0(A \times T/T)$, there exists a unique morphism $f \colon T \to A^t$ such that $(\mathrm{id} \times f)^*\mathcal{P} = P \in \mathrm{Pic}^0(A \times T/T)$.
- For all varieties T over k and all $f, g \colon T \to A^t$, we have

$$\big(\mathrm{id} \times (f+g)\big)^*\mathcal{P} = (\mathrm{id} \times f)^*\mathcal{P} + (\mathrm{id} \times g)^*\mathcal{P}.$$

If the dual exists, it is unique up to a unique isomorphism.

Theorem 18 (Grothendieck-Murre)**.** *Dual abelian varieties exist.*

1.3 Duality theory

Let $f \colon A \to B$ be a morphism of abelian varieties over k. (By the Rigidity Lemma, this is equivalent to saying that f is a morphism of varieties such that $f(0) = 0$.) Let $(A^t, \mathcal{P})$, $(B^t, \mathcal{Q})$ be the duals of A and B, respectively.

Consider the pullback $(f \times \mathrm{id}_{B^t})^*\mathcal{Q}$ along the map $f \times \mathrm{id}_{B^t} \colon A \times B^t \to B \times B^t$. Note that we have the following commutative diagram.

$$\begin{array}{ccc} A \times B^t & \xrightarrow{f \times \mathrm{id}_{B^t}} & B \times B^t \\ {\scriptstyle \mathrm{id}_A \times 0}\uparrow & & \uparrow{\scriptstyle \mathrm{id}_B \times 0} \\ A \times B^t & \xrightarrow{f \times \mathrm{id}_{B^t}} & B \times B^t \end{array}$$

It shows that $(\mathrm{id}_A \times 0)^*(f \times \mathrm{id}_{B^t})^*Q = (f \times \mathrm{id}_{B^t})^*(\mathrm{id}_B \times 0)^*Q = 0$, hence $(f \times \mathrm{id}_{B^t})^*Q$ is algebraically equivalent to 0.

Hence there exists, by the universal property of $(A^t, \mathcal{P})$, a unique morphism $f^t \colon B^t \to A^t$ such that $(f \times \mathrm{id}_{B^t})^*\mathcal{Q} = (\mathrm{id}_A \times f^t)^*\mathcal{P}$, which is called the *dual morphism of f*.

Now let $(A^t, \mathcal{P})$ be the dual abelian variety of A. So $(0 \times \mathrm{id}_{A^t})^*\mathcal{P}$ is trivial by definition, and as $(\mathrm{id}_A \times 0)^*\mathcal{P}$ corresponds to $0 \in A^t(k)$, it is trivial as well. Hence for $\sigma \colon A^t \times A \to A \times A^t$, we find that $\sigma^*\mathcal{P}$ is algebraically equivalent to 0 as well.

Now let $(A^{tt}, \mathcal{P}^t)$ be a dual of A^t. By the universal property of A^{tt}, $\sigma^*\mathcal{P}$ defines a morphism $\kappa_A \colon A \to A^{tt}$.

Theorem 19. *Let A be an abelian variety over k. Then κ_A is an isomorphism (*double duality*) and $\kappa_{A^t} = \kappa_A^{-t}$ (*triple duality*).*

From this point on, we will identify A with A^{tt} via κ_A.

1.4 Polarisations

Our identification allows us to talk about *symmetric* morphisms $A \to A^t$.

Definition 20. Let A be an abelian variety over k, and let A^t be its dual. A morphism $f \colon A \to A^t$ is *symmetric* if $f = f^t$.

Definition 21. Let A be an abelian variety over k, and let $(A^t, \mathcal{P})$ be its dual. A *polarisation* of A is a symmetric isogeny $\rho \colon A \to A^t$ such that $(\mathrm{id}, \rho)^*\mathcal{P}$ is ample. Note that (id, ρ) is a map $A \to A \times A^t$. A polarisation ρ of A is *principal* if ρ has degree 1 (so ρ is an isomorphism).

In the remainder of this section, we will construct examples of polarisations.

Let T be a variety over k, and let $P \in \operatorname{Pic}(A \times T/T)$. Consider $\mu \times \mathrm{id}_T, \pi_1 \times \mathrm{id}_T \colon A \times A \times T \to A \times T$. Then we can consider the *Mumford map* (or *universal translation map*)

$$\mathcal{S}\colon \operatorname{Pic}(A \times T/T) \to \operatorname{Pic}(A \times A \times T/A \times T),\ P \mapsto (\mu \times \mathrm{id})^* P - (\pi_1 \times \mathrm{id})^* P.$$

Theorem 22. *Let A be an abelian variety over k, and let T be a variety over k. Then* $\ker \mathcal{S} = \operatorname{Pic}^0(A \times T/T)$.

Let us now consider the case that $T = \mathbb{A}^0_k$. Then $\mathcal{S}(P)$ defines a morphism $\varphi(P)\colon A \to A^t$ by the universal property of A^t. For all varieties T over k, it is given by

$$\operatorname{Hom}(T, A) \to \operatorname{Pic}^0(A \times T/T),\ t \mapsto \tau_t^* P_T - P_T,$$

where $\tau_t\colon A \times T \to A \times T$ is the *translation-by-t* map $(x, y) \mapsto (x + ty, y)$, and P_T is the pullback of P along the projection $A \times T \to A$.

In particular, if we take $0 \in A(k)$, then we see that $\varphi(P)(0) = 0$, so by the Rigidity Lemma, $\varphi(P)$ is in fact a morphism of abelian varieties. Moreover, note that φ defines a group morphism $\operatorname{Pic} A \to \operatorname{Hom}_{\mathrm{AV}}(A, A^t)$ (as $\mathcal{S}$ defines a group morphism $\operatorname{Pic} A \to \operatorname{Pic}(A \times A/A)$), so by Theorem 22, we obtain an exact sequence

$$0 \longrightarrow A^t(k) \longrightarrow \operatorname{Pic} A \xrightarrow{\ \varphi\ } \operatorname{Hom}_{\mathrm{AV}}(A, A^t)$$

of abelian groups.

We give some properties of φ without proof.

Theorem 23. *Let A be an abelian variety over k, and let $P \in \operatorname{Pic} A$. Then the following are true.*

- *$\varphi(P)$ is symmetric.*
- *$\varphi(P)$ is an isogeny if and only if P is ample.*

So all $\varphi(P)$ with P ample are polarisations. We also have a converse.

Theorem 24. *Let $f\colon A \to A^t$ be a polarisation. Then there exists a finite separable field extension K of k, and $P_K \in \operatorname{Pic} A_K$ ample such that $f_K = \varphi(P_K)$.*

Definition 25. A *polarised abelian variety* is a pair (A, ρ) of an abelian variety A and a polarisation $\rho\colon A \to A^t$. A *morphism* between two polarised abelian varieties (A, ρ) and (A', ρ') is a morphism $f\colon A \to A'$ of abelian varieties such that $\rho = f^t \rho' f$.

Fact 26. *Let (A, ρ) be a polarised abelian variety over k.*

- *(A, ρ) has finitely many automorphisms.*
- *Let $n \geq 3$ be an integer. Then (A, ρ, α), where $\alpha\colon (\mathbb{Z}/n\mathbb{Z})^{2 \dim A} \to A(k)[n]$ is an isomorphism, has no non-trivial automorphisms.*

In fact, it turns out that moduli spaces of polarised abelian varieties, together with a full level n structure, exist if $n \geq 3$.

1.5 Polarisations of elliptic curves

Let E be an elliptic curve over k^{sep}. We first study E^t. Consider $\mathcal{P} = \langle 0 \times E\rangle - \langle \Delta \rangle \in \operatorname{Pic}(E \times E/E)$. Its pullback along $\mathrm{id}_E \times 0$ is trivial, hence $\mathcal{P}$ is algebraically equivalent to 0, so it defines a morphism $f\colon E \to E^t$. This is an isomorphism by [K-Ma, Sect 2.1]. Hence $(E, \mathcal{P})$ is the dual of E, and $E^t(k^{\mathrm{sep}}) = \operatorname{Pic}^0 E$, which here is the subgroup of $\operatorname{Pic} E$ of elements of degree 0.

Now note that by Theorem 24, all polarisations of E are of the form $\varphi(P)$ with $P \in \operatorname{Pic} E$ of positive degree d (which, by Riemann-Roch is equivalent to P being ample). The exact sequence

$$0 \longrightarrow \operatorname{Pic}^0 E \longrightarrow \operatorname{Pic} E \xrightarrow{\varphi} \operatorname{Hom}_{\mathrm{AV}}(E, E)$$

induces a morphism $\varphi\colon \operatorname{Pic} E / \operatorname{Pic}^0 E = \mathbb{Z} \to \operatorname{End}_{\mathrm{AV}}(E)$ of abelian groups. Hence, to understand all polarisations of E, we only need to understand $\varphi(1) = \varphi(\langle 0 \rangle)$.

By the description given earlier, $\varphi(1)$ is given on $\bar{k}$-points by $x \mapsto \langle -x \rangle - \langle 0 \rangle = \langle 0 \rangle - \langle x \rangle = x$, hence $\varphi(1) = \mathrm{id}_E$. It follows that $\varphi(d)$ is the multiplication-by-d map on E for all positive d, which has degree d^2.

2 Polarisations, complex analytically

Main references: [Mi], [Mu]

2.1 Line bundles and Riemann forms on complex tori

Definition 27. Let V be a finite-dimensional complex vector space.

A *Hermitian form* on V is a map $H\colon V \times V \to \mathbb{C}$ with the following properties.

- For all $z \in \mathbb{C}$, $v, w \in V$, we have $H(zv, w) = zH(v, w) = H(v, \bar{z}w)$. ($H$ is linear in the first factor, and *anti-linear* in the second factor, i.e. H is a *sesquilinear form*.)
- For all $v, w \in V$, we have $H(v, w) = \overline{H(w, v)}$. ($H$ is *symmetric*.)

A Hermitian form H on V is *positive definite* if $H(v, v) > 0$ for all $v \in V$.

A *pre-Riemann form* on V is a map $E\colon V \times V \to \mathbb{R}$ with the following properties.

- E is $\mathbb{R}$-bilinear.
- E is skew-symmetric (i.e. for all $v, w \in V$, we have $E(v, w) = -E(w, v)$).
- For all $v, w \in V$, we have $E(iv, iw) = E(v, w)$.

These two notions are related in the following way.

Lemma 28. *Let V be a finite-dimensional complex vector space. Let $\mathcal{H}_V$ denote the set of Hermitian forms on V, and let $\mathcal{E}_V$ denote the set of pre-Riemann forms on V. Then the maps $\mathcal{H}_V \to \mathcal{E}_V$, $H \mapsto \operatorname{im} H$ and $\mathcal{E}_V \to \mathcal{H}_V$, $E \mapsto \big((v, w) \mapsto E(iv, w) + iE(v, w)\big)$ are inverses.*

This allows us to define the following.

Definition 29. Let V be a complex vector space, and let $\Lambda \subseteq V$ be a lattice. A *pseudo-Riemann form* with respect to Λ is a pre-Riemann form E such that $E(\Lambda \times \Lambda) \subseteq \mathbb{Z}$. A *Riemann form* with respect to Λ is a pseudo-Riemann form that is positive definite.

If the complex torus V/Λ admits a Riemann form, we say that it is *polarisable*. Complex tori of dimension at least 2 are usually not polarisable.

Now fix a pseudo-Riemann form E, and let H be its associated Hermitian form.

Definition 30. An *E-character* is a map $\alpha\colon \Lambda \to \mathbb{C}_1^* = \{z \in \mathbb{C} : |z| = 1\}$ such that

$$\alpha(\lambda_1 + \lambda_2) = e^{\pi i E(\lambda_1, \lambda_2)} \alpha(\lambda_1) \alpha(\lambda_2).$$

For any fixed E, there exist E-characters.

Let G be the set of pairs (E, α), with E a pseudo-Riemann form on V, and α an E-character. Then G carries the structure of an abelian group with operation given by $(E_1, \alpha_1) + (E_2, \alpha_2) = (E_1 + E_2, \alpha_1 \alpha_2)$, and neutral element $(0, 1)$.

Given any $(E, \alpha) \in G$, we can construct a (classical) line bundle on V/Λ as follows. Let Λ act on V via translations, and let Λ act on $V \times \mathbb{C}$ by

$$(v, z)\lambda = \left(v + \lambda, e^{\pi H(v,\lambda) + \frac{1}{2}\pi H(\lambda,\lambda)}\alpha(\lambda)z\right).$$

This defines a Λ-equivariant map $V \times \mathbb{C} \to V$, hence a classical line bundle $(V \times \mathbb{C})/\Lambda \to V/\Lambda$ on V/Λ, and we can construct a line bundle $\mathcal{L}(E, \alpha)$ from this by taking the sheaf of its local sections (in the category of complex manifolds).

Theorem 31 (Appell-Humbert). *The map $G \to \operatorname{Pic} X$, $(E, \alpha) \mapsto \mathcal{L}(E, \alpha)$ is an isomorphism of groups.*

2.2 The dual torus

Let V be a complex finite-dimensional vector space, and let $\Lambda \subseteq V$ be a lattice. Denote by V^t the set of anti-linear maps $V \to \mathbb{C}$. Let $\Lambda^t = \left\{f \in V^t : \operatorname{im} f(\Lambda) \subseteq \mathbb{Z}\right\}$. Then V^t is a complex vector space of the same dimension as V, and Λ^t is a lattice inside V^t. We say that V^t/Λ^t is the *dual torus* of V/Λ. It always admits a Riemann form, unlike general complex tori.

To draw an analogy with the case of an abelian variety over $\mathbb{C}$, note that we can attach to an $f \in V^t$, an element $(0, e^{2\pi i \operatorname{im} f(\cdot)})$ of G. This defines a group morphism $V^t/\Lambda^t \to G$ with image consisting of the (H, α) with $H = 0$. Hence we can identify V^t/Λ^t with the set of line bundles on V/Λ that are *algebraically equivalent to* 0.

A pseudo-Riemann form E on V defines, via its associated Hermitian form H, a map $V \to V^t$ via $v \mapsto H(v, \cdot) \in V^t$. As $H(v, \cdot) \in \Lambda^t$ if and only if $E(v, \Lambda) \subseteq \mathbb{Z}$, it follows that it induces a map $\varphi(E)\colon V/\Lambda \to V^t/\Lambda^t$. If E is a Riemann form, then $\varphi(E)$ is in fact an isogeny, and we call $\varphi(E)$ a *polarisation* of V/Λ. In fact, we can view φ as a group morphism $G \to \operatorname{Hom}(V/\Lambda, V^t/\Lambda^t)$ with kernel V^t/Λ^t.

We have the following theorem, comparing the algebraic and complex analytic sides.

Theorem 32. *There exists an equivalence between the category of abelian varieties over $\mathbb{C}$ and the category of polarisable complex tori, called the* analytification functor*. Under this equivalence,*

- *duals of abelian varieties correspond to duals of complex tori;*
- *polarisations of abelian varieties correspond to polarisations of complex tori.*

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