Specialisation theory for the étale fundamental group

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Rings will always be associative, commutative, and unital.

The purpose of this talk is to bridge the gap between the characteristic 0 case, and the positive characteristic case. The following construction will be crucial for this.

1 The Witt scheme

The main reference for this section is [R].

1.1 Definition

To do this in the most general setting, we first give the following notion.

Definition 1. A *divisor-stable set* is a subset $P \subseteq \mathbb{Z}_{>0}$ such that for all $n \in P$ and $d \in \mathbb{Z}_{>0}$ such that $d \mid n$, we have $d \in P$.

The most important examples for us will be the sets $P_{p,n} = \{1, p, \ldots, p^n\}$ and $P_p = \bigcup_{n=1}^{\infty} P_{p,n}$, where p is a prime number, though the case $P = \mathbb{Z}_{>0}$ is interesting in its own right as well.

From now on, fix a divisor-stable set P.

Definition 2. Let $n \in P$. Then the *n-th Witt polynomial* with respect to P is

$$w_n = \sum_{d \mid n} d X_d^{n/d} \in \mathbb{Z}[X_i : i \in P].$$

An important remark is that the leading term of w_n is nX_n, so we have the equality $nX_n = w_n - \sum_{d \mid n, d \neq n} d X_d^{n/d}$.

Definition 3. A *Witt functor* is a functor $W_P \colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Ring}$ with the following properties.

- $F \circ W_P = F \circ \mathbb{A}^P$, where $F \colon \mathrm{Ring} \to \mathrm{Set}$ is the forgetful functor, and $\mathbb{A}^P \colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Ring}$ is the functor $S \mapsto \mathcal{O}_S(S)^P$.
- $(w_n)_{n \in P}$ defines a natural transformation $W_P \to \mathbb{A}^P$ (of functors $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Ring}$).

More concretely, the conditions mean that $W_P(S) = \mathbb{A}^P(S)$ $(= \mathcal{O}_S(S)^P)$ as sets, functorial in S, and that $w_n \colon W_P(S) \to \mathcal{O}_S(S)$ is a morphism of rings for all $n \in P$.

Proposition 4. *Witt functors are unique.*

Proof. Let W_P be a Witt functor, and let $\mathcal{R} = \mathbb{Z}[A_n, B_n : n \in P]$. As no element in P is a zero-divisor in $\mathcal{R}$, the maps $w_n \colon W_P(\mathcal{R}) \to \mathcal{R}$ are injective, hence so is the map $(w_n)_{n \in P} \colon W_P(\mathcal{R}) \to \mathcal{R}^P$. This fixes the ring structure on $W_P(\mathcal{R})$.

Now we use this to fix the ring structure on $W_P(S)$ for any scheme S. Let $a, b \in W_P(S)$. Consider the map $\mathcal{R} \to \mathcal{O}_S(S)$ sending A_n to a_n and B_n to b_n for all $n \in P$, and the corresponding map of schemes $S \to \operatorname{Spec} \mathcal{R}$. It induces the morphism $W_P(\mathcal{R}) \to W_P(S)$ of rings, which determines $a + b$ and ab in $W_P(S)$. □

A similar argument shows that the additive unit in any $W_P(S)$ is $(0, 0, \ldots)$ and that the multiplicative unit in any $W_P(S)$ is $(1, 0, 0, \ldots)$.

Theorem 5. *There exists a Witt functor.*

Proof. For a construction, see [R, Sect. 3]. □

So from now on, we let W_P be *the* Witt functor $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Ring}$. By definition, W_P is representable by an affine ring scheme whose underlying scheme is $(\mathbb{A}^1)^P$. We denote this ring scheme (called the *Witt scheme*) by W_P as well. If P is clear from the context, we will often drop P from the notation.

For any scheme S, and $x \in W(S)$, the coordinates of x are called the *Witt components* of x, and the $w_n(x)$ are called the *ghost components* of x.

1.2 The "classical" case

In the remainder of this section, we fix a prime p. Let $P = \{p^i : i \in \mathbb{Z}_{\geq 0}\}$. (Note that our notation in this case is non-standard, compared to other treatments of Witt vectors.)

Theorem 6. *Let K be a perfect ring of characteristic p (i.e. $x \mapsto x^p$ is bijective). Then $W(K)$ is the (up to isomorphism) unique ring that is Hausdorff and complete with respect to the p-adic topology, such that p is regular in $W(K)$, and such that $W(K)/pW(K) = K$. (In other words, $W(K)$ is the unique* strict p-ring *with* residue ring *K.)*

Corollary 7. *$W(\mathbb{F}_p) = \mathbb{Z}_p$, and $W(\mathbb{F}_{p^n})$ is the unique unramified extension of $\mathbb{Z}_p$ of degree n. In general, if k is a field of characteristic p, then $W(k)$ is a discrete valuation ring.*

Example 8. Let $P = \{1, p\}$. We want to describe for this P the ring operations on W. As seen in the proof of Proposition 4, it suffices to do so for $W(\mathcal{R})$, where $\mathcal{R} = \mathbb{Z}[A_1, A_p, B_1, B_p]$, and the two elements $A = (A_1, A_p)$, $B = (B_1, B_p)$ of $W(\mathcal{R})$.

For the addition, we note the following.

- From $w_1 = X_1$ and $w_1(A + B) = w_1(A) + w_1(B)$, it immediately follows that
$$(A + B)_1 = A_1 + B_1.$$
- As $w_p = X_1^p + pX_p$, we see that
$$(A + B)_1^p + p(A + B)_p = w_p(A + B) = w_p(A) + w_p(B) = A_1^p + B_1^p + pA_p + pB_p,$$
hence
$$\begin{aligned}(A + B)_p &= A_p + B_p - \tfrac{1}{p}\big((A_1 + B_1)^p - A_1^p - B_1^p\big)\\ &= A_p + B_p - \sum_{n=1}^{p-1} \tfrac{1}{p}\tbinom{p}{n} A_1^n B_1^{p-n}.\end{aligned}$$

For the multiplication, in the same way, we see that
$$(AB)_1 = A_1B_1 \qquad (AB)_p = A_1^p B_p + A_p B_1^p + pA_p B_p.$$

2 Topological finite generatedness for curves

2.1 Étale fundamental groups and morphisms of schemes

No proofs will be given, for proofs (in the more general setting of functors between Galois categories), see [SGA1, Exp. V.6].

Let $f\colon X \to S$ be a morphism of schemes, let $\overline{x}$ be a geometric point of X, and let $\overline{s}$ be its image in S. (We will also say that X and S are *geometrically pointed schemes*, and that f is a *morphism of geometrically pointed schemes*.) Then we obtain a morphism $\pi_1(f)\colon \pi_1(X, \overline{x}) \to \pi_1(S, \overline{s})$ on étale fundamental groups, as follows. Let $(\sigma_{X'/X})_{X'/X}$ be an automorphism of the fibre functor $\mathrm{FEt}_X \to \mathrm{FSet}$ with respect to $\overline{x}$. Then put $\pi_1(f)(\sigma_{X'/X}) = (\sigma_{S' \times_S X/X})_{S'/S}$, which is an automorphism of the fibre functor $\mathrm{FEt}_S \to \mathrm{FSet}$ with respect to $\overline{s}$.

We want to describe exactness of sequences of étale fundamental groups in terms of the morphisms themselves. The main properties with respect to this are the following ones.

Proposition 9. *Let $f\colon X \to S$ be a morphism of geometrically pointed schemes.*

- *$\pi_1(f)$ is injective if and only if for all finite étale covers X'/X, there exists a finite étale cover S'/S and a morphism from a connected component of $S' \times_S X$ to X'.*
- *$\pi_1(f)$ is surjective if and only if for all connected finite étale covers S'/S, $S' \times_S X/X$ is connected, or equivalently, if and only if $- \times_S X\colon \mathrm{FEt}_S \to \mathrm{FEt}_X$ is fully faithful.*
- *$\pi_1(f) = 1$ if and only if for all finite étale covers S'/S, $S' \times_S X/X$ is the trivial cover.*

Moreover, let $g\colon Y \to X$ be a morphism of geometrically pointed schemes.

- *$\operatorname{im} \pi_1(g) \subseteq \ker \pi_1(f)$ if and only if for all finite étale covers X'/X such that $X' \times_X Y/Y$ admits a section, there exists a finite étale cover S'/S and a morphism from a connected component of $S' \times_S X/X$ to X'.*

2.2 An exact sequence of étale fundamental groups

Reference for the remainder: [SGA1, Exp. X]

Let $f\colon X \to S$ be a qcqs morphism of schemes, so that $f_*\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_S$-algebra. Then we have a canonical map $c\colon X \to \operatorname{Spec}_S f_*\mathcal{O}_X$ of S-schemes.

Proposition 10 (Stein factorisation, [S, 03GX], [EGA1, III.7.8.10])**.** *Let $f\colon X \to S$ be a proper morphism of schemes. Then the canonical map $c\colon X \to \operatorname{Spec}_S f_*\mathcal{O}_X$ is proper, and its geometric fibres are connected. Moreover, $\operatorname{Spec}_S f_*\mathcal{O}_X$ is integral over S and is the normalisation of S in X.*

If S is locally noetherian, then $\operatorname{Spec}_S f_*\mathcal{O}_X$ is finite over S. If in addition, f is flat and its geometric fibres are reduced, then $\operatorname{Spec}_S f_*\mathcal{O}_X$ is finite étale over S.

Theorem 11. *Let $f\colon X \to S$ be a proper, flat morphism of schemes, of which the geometric fibres are reduced, where S is locally noetherian and connected, and suppose that $f_*\mathcal{O}_X = \mathcal{O}_S$ (so that by Stein factorisation, the geometric fibres of f are connected). Let $\overline{s}$ be a geometric point of S.*

Let X' be a finite étale connected cover of X. Then there exists a finite étale cover S' of S and an isomorphism $X' \to X \times_S S'$ of X-schemes, if and only if $X'_{\overline{s}} \to X_{\overline{s}}$ admits a section.

Proof. First suppose that there exists a finite étale cover S' of S and an isomorphism $X' \to X \times_S S'$ of X-schemes. This induces an isomorphism $X'_{\overline{s}} \to X_{\overline{s}} \times S'_{\overline{s}}$, which hence induces a section of $X'_{\overline{s}} \to X_{\overline{s}}$.

Now suppose that $X'_{\overline{s}} \to X_{\overline{s}}$ admits a section. Let h denote the morphism $X' \to X$, and put $S' = \operatorname{Spec}_S f_*h_*\mathcal{O}_{X'}$. Consider the morphism $\varphi = (h, fh)\colon X' \to X \times_S S'$. By Stein factorisation, S'/S is finite étale, so $X \times_S S'/X$ is finite étale, hence we deduce that φ is finite étale as well. Moreover, again by Stein factorisation, S' is connected as fh is surjective and X' is connected. As $X \times_S S'/S'$ is proper and has connected fibres, it follows that $X \times_S S'$ is connected. Hence α

has constant positive degree, and it is an isomorphism if and only if its degree is 1, which can be checked at any geometric point of $X \times_S S'$.

Consider $\alpha_{\bar{s}} \colon X'_{\bar{s}} \to X_{\bar{s}} \times S'_{\bar{s}}$. Then note that $\pi_{S'_{\bar{s}}} \alpha_{\bar{s}} \colon X'_{\bar{s}} \to S'_{\bar{s}}$ has connected fibres by Stein factorisation, so $X'_{\bar{s}}$ consists of $\# S'_{\bar{s}}$ connected components. As $\alpha_{\bar{s}}$ is surjective, it induces a bijection between the sets of connected components of $X'_{\bar{s}}$ and those of $X_{\bar{s}} \times S'_{\bar{s}}$. Now a section of $X'_{\bar{s}} \to X_{\bar{s}}$ defines an element σ of $S'_{\bar{s}}$, and by the above, the degree of $\alpha_{\bar{s}}$ on any geometric point $\bar{t}$ on $X_{\bar{s}} \times \{\sigma\}$ is equal to 1, as desired. □

Corollary 12. *Let $f \colon X \to S$ and $\bar{s}$ be as in Theorem 11. Moreover, let $\bar{a}$ be a geometric point of $X_{\bar{s}}$, and $\bar{b}$ and $\bar{c}$ be its image in X and S, respectively. Then the following sequence of groups is exact.*

$$\pi_1(X_{\bar{s}}, \bar{a}) \longrightarrow \pi_1(X, \bar{b}) \longrightarrow \pi_1(S, \bar{c}) \longrightarrow 1$$

We will use the following property, without proof.

Proposition 13 ([SGA I, Exp. X.1.8])**.** *Let k be an algebraically closed field X. Let S be a proper, connected k-scheme, and let K be an algebraically closed field extension of k. Let $\bar{s}_K$ be a geometric point of S_K, and let $\bar{s}$ be its image in S. Then the map $\pi_1(S_K, \bar{s}_K) \to \pi_1(S, \bar{s})$ is an isomorphism.*

Corollary 14. *Let k be an algebraically closed field of characteristic 0, and let $g \in \mathbb{Z}_{\geq 0}$. Let S be a smooth proper connected curve over k of genus g, and let $\bar{s}$ be a geometric point on S. Then $\pi_1(S, \bar{s})$ is isomorphic to the quotient of the free profinite group on $2g$ generators $s_1, t_1, \ldots, s_g, t_g$ by its subgroup generated by*

$$(s_1 t_1 s_1^{-1} t_1^{-1}) \ldots (s_g t_g s_g^{-1} t_g^{-1}).$$

Proof. For the (profinite completion of) the topological fundamental group of a Riemann surface, this is classical. By the Riemann Existence Theorem, we get the result in case $k = \mathbb{C}$.

We will now reduce the general case to this one. As S is of finite type over k, there exists an algebraically closed field k_0 of finite transcendence degree over $\mathbb{Q}$, and a k_0-scheme S_0 such that $S = S_0 \times_{k_0} \operatorname{Spec} k$. Note that S_0 is again a smooth proper connected curve over k_0. Let $\bar{s}_0$ be the image of $\bar{s}$ in S_0. By Proposition 13, it follows that $\pi_1(S, \bar{s}) \to \pi_1(S_0, \bar{s}_0)$ is an isomorphism.

Now note that we have an embedding $k_0 \to \mathbb{C}$, as k_0 has finite transcendence degree over $\mathbb{Q}$. Let $S' = S_0 \times_{k_0} \operatorname{Spec} \mathbb{C}$, let $\bar{s}'$ be a geometric point of S', and let $\bar{s}'_0$ be its image in S_0. Again by Proposition 13, $\pi_1(S', \bar{s}') \to \pi_1(S_0, \bar{s}'_0)$ is an isomorphism. Hence (as the isomorphism class of the étale fundamental group does not depend on the chosen base point) $\pi_1(S', \bar{s}') \cong \pi_1(S, \bar{s})$, as desired. □

2.3 Specialisation theory

We give the following theorems without proof.

Theorem 15 ([EGA III, Ch. 5])**.** *Let S be the spectrum of a local complete noetherian ring, X a proper S-scheme. Let X_0 be the closed fibre of X, let $\bar{x}_0$ be a geometric point of X_0, and let $\bar{x}$ be its image in X. Then the morphism $\pi_1(X_0, \bar{x}_0) \to \pi_1(X, \bar{x})$ is an isomorphism.*

Theorem 16 ([SGA I, Exp. III.7])**.** *Let A be a complete local ring with residue field k. Let S_0 be a smooth proper curve over k of genus g. Then there exists a smooth proper curve S over A of genus g such that S_0 is the special fibre of S.*

Recall here that a *curve of genus g* over a scheme S is a flat morphism of finite presentation such that all geometric fibres are (irreducible) curves of genus g.

Theorem 17. *Let S be the spectrum of a local artinian ring with residue field k. Let X be an S-scheme, and let $X_{\overline{0}} = X \times_S \operatorname{Spec} \overline{k}$. Let $\overline{x}_{\overline{0}}$ be a geometric point of $X_{\overline{0}}$, and let $\overline{x}, \overline{s}, \overline{t}$ be their images in X, S, and $\operatorname{Spec} k$ respectively.*

Suppose that $X_{\overline{0}}$ is quasi-compact and connected. Then the sequence

$$1 \longrightarrow \pi_1(X_{\overline{0}}, \overline{x}_{\overline{0}}) \longrightarrow \pi_1(X, \overline{x}) \longrightarrow \pi_1(S, \overline{s}) \longrightarrow 1$$

is exact, and we have an isomorphism $\pi_1(\operatorname{Spec} k, \overline{t}) \to \pi_1(S, \overline{s})$.

Sketch of proof. As $\operatorname{Spec} k$ is the reduced subscheme of S, we have an isomorphism $\pi_1(\operatorname{Spec} k, \overline{t}) \to \pi_1(S, \overline{s})$, so we may reduce immediately to $S = \operatorname{Spec} k$.

Now let k' be the maximal purely inseparable extension of k, and let $S' = \operatorname{Spec} k'$, $X' = X \times_S S'$. Then we have a sequence

$$1 \longrightarrow \pi_1(X_{\overline{0}}, \overline{x}_{\overline{0}}) \longrightarrow \pi_1(X', \overline{x}) \longrightarrow \pi_1(S', \overline{s}) \longrightarrow 1,$$

which is isomorphic to the one we're considering, as faithfully flat, quasi-compact, purely inseparable morphisms are *morphismes de descente effective* for the "fibred category" of étale separated morphisms of finite type.

Hence we are reduced to the case that k is perfect. Let for all finite Galois sub-extensions l of $\overline{k}/k$, $X_l = X \times_k \operatorname{Spec} l$, and let $\overline{x}_l$ be the image of $\overline{x}_{\overline{0}}$ in X_l. Then first note that the morphism $\pi_1(X_{\overline{0}}, \overline{x}_{\overline{0}}) \to \lim_l \pi_1(X_l, \overline{x}_l)$ is surjective, as $X_{\overline{0}}$ is connected, and injective, as every finite étale cover arises from some finite Galois extension of k, hence an isomorphism.

Now note that the functor $\mathrm{FEt}_S \to \mathrm{FEt}_X$, $S' \mapsto S' \times_S X$ is fully faithful, so $\operatorname{Aut}_S(\operatorname{Spec} l) = \operatorname{Aut}_X(X_l)$. As $\operatorname{Gal}(l/k) = \operatorname{Gal}(k'/k)/\operatorname{Gal}(k'/l)$ and $\operatorname{Aut}_X(X_l) = \pi_1(X, \overline{x})/\pi_1(X_l, \overline{x}_l)$, we obtain an exact sequence

$$1 \longrightarrow \pi_1(X_l, \overline{x}_l) \longrightarrow \pi_1(X, \overline{x}) \longrightarrow \operatorname{Gal}(l/k) \longrightarrow 1,$$

which gives us the desired exact sequence after passing to the limit. □

Now let S be the spectrum of a local complete noetherian ring with residue field k, and let X a proper S-scheme. Let X_0 be the closed fibre of X, let $X_{\overline{0}} = X_0 \times_k \operatorname{Spec} \overline{k}$, and let $\overline{x}_{\overline{0}}$ be a geometric point of $X_{\overline{0}}$, and let $\overline{x}_0$, $\overline{x}$, $\overline{s}$, $\overline{t}$ be its image in X_0, X, S, and $\operatorname{Spec} k$, respectively. Then by the above, we have a short exact sequence

$$1 \longrightarrow \pi_1(X_{\overline{0}}, \overline{x}_{\overline{0}}) \longrightarrow \pi_1(X_0, \overline{x}_0) \longrightarrow \pi_1(\operatorname{Spec} k, \overline{t}) \longrightarrow 1.$$

By Theorem 15, we have isomorphisms $\pi_1(X_0, \overline{x}_0) \to \pi_1(X, \overline{x})$ and $\pi_1(\operatorname{Spec} k, \overline{t}) \to \pi_1(S, \overline{s})$. This gives us a short exact sequence

$$1 \longrightarrow \pi_1(X_{\overline{0}}, \overline{x}_{\overline{0}}) \longrightarrow \pi_1(X, \overline{x}) \longrightarrow \pi_1(S, \overline{s}) \longrightarrow 1.$$

Now let s_1 be any point of S, with residue field K. Let X_1 be the fibre of X above s_1, let $X_{\overline{1}} = X_1 \times_K \operatorname{Spec} \overline{K}$ the corresponding geometric fibre. Let $\overline{x}_{\overline{1}}$ be any geometric point on $X_{\overline{1}}$, and let $\overline{x}'$, $\overline{s}'$ be its image in X, S, respectively. We then have a complex of groups

$$\pi_1(X_{\overline{1}}, \overline{x}_{\overline{1}}) \longrightarrow \pi_1(X, \overline{x}') \longrightarrow \pi_1(S, \overline{s}') \longrightarrow 1,$$

hence induces, *up to inner automorphisms*, a morphism $\pi_1(X_{\bar{1}}, \bar{x}_{\bar{1}}) \to \pi_1(X_{\bar{0}}, \bar{x}_{\bar{0}})$, called the *specialisation morphism*. If this complex is exact, then the specialisation morphism is in fact surjective. By Corollary 12, this is the case if we assume that $f\colon X \to S$ is a proper flat morphism with geometrically reduced fibres, and $f_*\mathcal{O}_X = \mathcal{O}_S$ (which by Stein factorisation amounts to saying that $X_{\bar{0}}$ is connected).

Corollary 18. *Let k be an algebraically closed field of positive characteristic, and let $g \in \mathbb{Z}_{\geq 0}$. Let S_0 be a smooth proper connected curve over k of genus g, and let $\bar{s}_0$ be a geometric point on S. Then $\pi_1(S_0, \bar{s}_0)$ is topologically generated by $2g$ generators $s_1, t_1, \ldots, s_g, t_g$ satisfying the relation*

$$(s_1 t_1 s_1^{-1} t_1^{-1}) \ldots (s_g t_g s_g^{-1} t_g^{-1}) = 1.$$

Proof. Let K be an algebraic closure of the field of fractions of the ring $W(k)$ of Witt vectors over k. Then by Theorem 16, there exists a proper smooth curve S of genus g over $W(k)$ such that S_0 is the closed fibre of S. Let $S_1 = S \times_{W(k)} \operatorname{Spec} K$, and let $\bar{s}_1$ be a geometric point of S_1. Then as S is proper and flat over $W(k)$, and has geometrically integral fibres, we have a surjective morphism $\pi_1(S_1, \bar{s}_1) \to \pi_1(S_0, \bar{s}_0)$, well-defined up to inner automorphisms. As S_1 is a proper smooth curve of genus g over K, the result now follows from Corollary 14. □

3 References

[R] J. Rabinoff, *The theory of Witt vectors*, `http://www.math.harvard.edu/~rabinoff/misc/witt.pdf`

[S] *Stacks Project*, `http://stacks.math.columbia.edu`