The dualising complex and Grothendieck-Serre duality

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1 Notation

For X a ringed space, we denote by $K(\mathcal{O}_X)$ the homotopy category of all $\mathcal{O}_X$-modules, and by $D(\mathcal{O}_X)$ the derived category of all $\mathcal{O}_X$-modules. If X is a scheme, then we denote by $K(\mathcal{Q}_X)$ the homotopy category of all quasi-coherent $\mathcal{O}_X$-modules, and by $D(\mathcal{Q}_X)$ the derived category of all quasi-coherent $\mathcal{O}_X$-modules. We will often use tacitly that for all qcqs schemes X, the category $D(\mathcal{Q}_X)$ is equivalent to $D_{qc}(\mathcal{O}_X)$. This is Corollary 5.5 of [1], with Remark 5.6 replaced by qcqs induction to obtain the stronger statement.

2 Preliminaries: Resolutions of unbounded complexes

The bulk from this section will be from the Stacks Project (with references to tags denoted [SPxxxx], where xxxx is the tag in question), which refers to Spaltenstein's article with the same name as this section. The main purpose of this section will be to state some results that we use as a black box (in the same way we use the existence of injective resolutions as a black box).

2.1 K-injective resolutions, right derived functors

We first recall the notion of a K-injective (right) resolution. See also [SP070G].

Definition 1 ([SP070I])**.** Let $\mathcal{A}$ be an abelian category. A complex $I^\cdot$ is *K-injective* if it satisfies one of the following equivalent conditions.

- For every acyclic complex $M^\cdot$, $\mathrm{Hom}_{K(\mathcal{A})}(M^\cdot, I^\cdot) = 0$.
- For every quasi-isomorphism $M^\cdot \to N^\cdot$ the map $\mathrm{Hom}_{K(\mathcal{A})}(N^\cdot, I^\cdot) \to \mathrm{Hom}_{K(\mathcal{A})}(M^\cdot, I^\cdot)$ is a bijection.
- For every complex $N^\cdot$ the map $\mathrm{Hom}_{K(\mathcal{A})}(N^\cdot, I^\cdot) \to \mathrm{Hom}_{D(\mathcal{A})}(N^\cdot, I^\cdot)$ is an isomorphism.

Definition 2. Let $\mathcal{A}$ be an abelian category, and let $A^\cdot$ be an object of $K(\mathcal{A})$. A *K-injective resolution* is a quasi-isomorphism $A^\cdot \to I^\cdot$ in $K(\mathcal{A})$ with $I^\cdot$ K-injective.

Theorem 3 (Spaltenstein [4, Lem. 4.3, Thm. 4.5])**.** *Let X be a ringed space. Then any complex $A^\cdot \in K(\mathcal{O}_X)$ admits a K-injective resolution of which each term is injective.*

Now given any additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories, and assume that every object of $K(\mathcal{A})$ admits a K-injective resolution. Then ([SP070K]) we have a right derived functor $RF\colon D(\mathcal{A}) \to D(\mathcal{B})$, and for any K-injective complex $I^\cdot$, we have $RFI^\cdot = FI^\cdot$.

We are most interested in the right derived functors $Rf_*\colon D(\mathcal{O}_X) \to D(\mathcal{O}_Y)$ for $f\colon X \to Y$ a morphism of ringed spaces, and $R\,\mathcal{H}om_{\mathcal{O}_X}(M^\cdot, -)\colon D(\mathcal{O}_X) \to D(\mathcal{O}_X)$ for X a ringed space.

2.2 K-flat resolutions, derived tensor products and pullbacks

Next, we introduce the notion of a K-flat (left) resolution. See [SP06Y7].

Definition 4. Let X be a ringed space. Let $M^\cdot, N^\cdot$ be two complexes in $K(\mathcal{O}_X)$. Then their *tensor product* is $\mathrm{Tot}(M^\cdot \otimes_{\mathcal{O}_X} N^\cdot)$.

This induces, for every complex $N^\cdot$ in $K(\mathcal{O}_X)$, a functor $K(\mathcal{O}_X) \to K(\mathcal{O}_X)$, $M^\cdot \mapsto \mathrm{Tot}(M^\cdot \otimes_{\mathcal{O}_X} N^\cdot)$, which is triangulated (see [SP06Y8]).

Definition 5. Let X be a ringed space. A complex $F^\cdot$ in $K(\mathcal{O}_X)$ is *K-flat* if it satisfies one of the following equivalent conditions.

- For any acyclic complex $M^\cdot$ in $K(\mathcal{O}_X)$, the complex $\mathrm{Tot}(M^\cdot \otimes_{\mathcal{O}_X} F^\cdot)$ is acyclic.
- For all $x \in X$ and any acyclic complex $M^\cdot_x$ in $K(\mathcal{O}_{X,x})$, the complex $\mathrm{Tot}(M^\cdot_x \otimes_{\mathcal{O}_{X,x}} F^\cdot_x)$ is acyclic.

Fact 6 ([SP079R], [SP06YC])**.** *Let $f\colon X \to Y$ be a morphism of ringed spaces. Let $A^\cdot, B^\cdot$ be K-flat complexes in $K(\mathcal{O}_Y)$. Then $\mathrm{Tot}(A^\cdot \otimes_{\mathcal{O}_Y} B^\cdot)$ and $f^*A^\cdot$ are K-flat complexes.*

Definition 7. Let X be a ringed space, and let $A^\cdot$ be an object of $K(\mathcal{O}_X)$. A *K-flat resolution* is a quasi-isomorphism $F^\cdot \to A^\cdot$ with $F^\cdot$ K-flat.

Theorem 8 ([SP06YF])**.** *Let X be a ringed space. Then any complex $A^\cdot \in K(\mathcal{O}_X)$ admits a K-flat resolution.*

Now let $N^\cdot$ be an object of $D(\mathcal{O}_X)$, and choose a K-flat resolution $F^\cdot \to N^\cdot$. Define the triangulated functor $K(\mathcal{O}_X) \to K(\mathcal{O}_X)$, $M^\cdot \mapsto \mathrm{Tot}(M^\cdot \otimes_{\mathcal{O}_X} F^\cdot)$. It sends quasi-isomorphisms to quasi-isomorphisms, since $F^\cdot$ is K-flat. Hence it induces a triangulated functor

$$- \overset{L}{\otimes}_{\mathcal{O}_X} N^\cdot \colon D(\mathcal{O}_X) \to D(\mathcal{O}_X)$$

between derived categories. This functor (up to isomorphism) does not depend on $F^\cdot \to N^\cdot$ [SP06YG].

We also have a triangulated functor

$$Lf^*\colon D(\mathcal{O}_Y) \to D(\mathcal{O}_X)$$

between derived categories, mapping $N^\cdot$ to $f^*F^\cdot$, where $F^\cdot \to N^\cdot$ is a K-flat resolution.

The following are easy exercises now.

Proposition 9. *Let $f\colon X \to Y$, and $g\colon Y \to Z$ be morphisms of ringed spaces. Then $Lf^*Lg^* = L(gf)^*$.*

Proposition 10. *Let $f\colon X \to Y$ be a morphism of ringed spaces, and let $M^\cdot, N^\cdot$ be objects of $D(\mathcal{O}_Y)$. Then*

$$Lf^*(M^\cdot \overset{L}{\otimes}_{\mathcal{O}_Y} N^\cdot) = Lf^*M^\cdot \overset{L}{\otimes}_{\mathcal{O}_X} Lf^*N^\cdot.$$

2.3 Weakly K-injective resolutions, derived pushforward, derived internal Hom

Definition 11. Let X be a ringed space. A complex $I^\cdot$ in $K(\mathcal{O}_X)$ is *weakly K-injective* if it satisfies one of the following equivalent conditions.

- For every K-flat acyclic complex $M^\cdot$, $\mathrm{Hom}_{K(\mathcal{O}_X)}(M^\cdot, I^\cdot) = 0$.
- For every quasi-isomorphism $M^\cdot \to N^\cdot$ with $M^\cdot, N^\cdot$ K-flat, the map $\mathrm{Hom}_{K(\mathcal{O}_X)}(N^\cdot, I^\cdot) \to \mathrm{Hom}_{K(\mathcal{O}_X)}(M^\cdot, I^\cdot)$ is an isomorphism.
- For every K-flat complex $N^\cdot$ the map $\mathrm{Hom}_{K(\mathcal{O}_X)}(N^\cdot, I^\cdot) \to \mathrm{Hom}_{D(\mathcal{O}_X)}(N^\cdot, I^\cdot)$ is an isomorphism.

Definition 12. Let X be a ringed space, and let $A^\cdot$ be a complex in $K(\mathcal{O}_X)$. A *weakly K-injective resolution* is a quasi-isomorphism $A^\cdot \to I^\cdot$ with $I^\cdot$ weakly K-injective.

Fact 13. *Let $f\colon X \to Y$ be a morphism of ringed spaces. Let $I^\cdot$ be a weakly K-injective complex in $K(\mathcal{O}_X)$. Then $f_*I^\cdot$ is a weakly K-injective complex in $K(\mathcal{O}_Y)$. Moreover, in $D(\mathcal{O}_Y)$, $Rf_*I^\cdot = f_*I^\cdot$.*

Fact 14. *Let X be a ringed space. Let $M^\cdot, N^\cdot$ be two complexes in $K(\mathcal{O}_X)$, such that either of the following holds.*

- *$N^\cdot$ is K-injective;*
- *$M^\cdot$ is K-flat and $N^\cdot$ is weakly K-injective.*

Then $\mathcal{H}om_{\mathcal{O}_X}(M^\cdot, N^\cdot)$ is weakly K-injective as well. Moreover, in $D(\mathcal{O}_X)$, we have $R\,\mathcal{H}om_{\mathcal{O}_X}(M^\cdot, N^\cdot) = \mathcal{H}om_{\mathcal{O}_X}(M^\cdot, N^\cdot)$.

This has the following nice consequence.

Corollary 15. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of ringed spaces. Then $Rg_*Rf_* = R(gf)_*$.*

Another consequence is the following, which we cannot formulate without the use of unbounded derived categories.

Corollary 16. *Let $f\colon X \to Y$ be a morphism of ringed spaces. Then (Lf^*, Rf_*) is adjoint pair of functors.*

Proof. Let $A^\cdot$ be an object of $D(\mathcal{O}_Y)$ and let $B^\cdot$ be an object of $D(\mathcal{O}_X)$. Choose a K-flat resolution $F^\cdot \to A^\cdot$ in $K(\mathcal{O}_Y)$, and a weakly K-injective resolution $B^\cdot \to I^\cdot$ in $K(\mathcal{O}_X)$. Then we have

$$\begin{aligned}\operatorname{Hom}_{D(\mathcal{O}_X)}(Lf^*A^\cdot, B^\cdot) &= \operatorname{Hom}_{D(\mathcal{O}_X)}(Lf^*F^\cdot, I^\cdot) = \operatorname{Hom}_{K(\mathcal{O}_X)}(f^*F^\cdot, I^\cdot) \\ &= \operatorname{Hom}_{K(\mathcal{O}_Y)}(F^\cdot, f_*I^\cdot) = \operatorname{Hom}_{D(\mathcal{O}_Y)}(F^\cdot, Rf_*I^\cdot) \\ &= \operatorname{Hom}_{D(\mathcal{O}_Y)}(A^\cdot, Rf_*B^\cdot),\end{aligned}$$

where we can move between the homotopy category and the derived category since both $I^\cdot$ and $f_*I^\cdot$ are weakly K-injective in their respective homotopy categories, and $f^*F^\cdot$ and $F^\cdot$ in their respective homotopy categories. □

Similarly, we have the following.

Corollary 17. *Let X be a ringed space, and let $A^\cdot$ be an object of $D(\mathcal{O}_X)$. Then $\left(- \overset{L}{\otimes}_{\mathcal{O}_X} A, R\,\mathcal{H}om_{\mathcal{O}_X}(A, -)\right)$ is an adjoint pair of functors.*

3 The dualising complex

Most of this section is Neeman [3].

3.1 Local theory in derived categories of schemes

First, we prove a lemma which allows us to check locally that a morphism in the derived category is an isomorphism.

Lemma 18. *Let X be a ringed space, and let $\mathcal{U}$ be an open cover of X. For $U \in \mathcal{U}$, let $j_U\colon U \to X$ denote the open immersion. Then the natural functor $D(\mathcal{O}_X) \to \prod_{U\in\mathcal{U}} D(\mathcal{O}_U)$, $A \mapsto (j_U^*A)_{U\in\mathcal{U}}$ is conservative.*

Proof. Note that for any ringed space Y, we have a conservative functor $H\colon D(\mathcal{O}_Y) \to \prod_{i\in\mathbb{Z}} \mathcal{O}_Y\text{-Mod}$ defined by cohomology. Hence we have a diagram

$$\begin{array}{ccc} D(\mathcal{O}_X) & \longrightarrow & \prod_{U\in\mathcal{U}} D(\mathcal{O}_U) \\ {\scriptstyle H}\big\downarrow & & \big\downarrow{\scriptstyle H} \\ \prod_{i\in\mathbb{Z}} \mathcal{O}_X\text{-Mod} & \longrightarrow & \prod_{i\in\mathbb{Z}, U\in\mathcal{U}} \mathcal{O}_U\text{-Mod} \end{array}$$

where the horizontal arrows are the natural ones. It commutes by locality of cohomology [SP01E1]. All arrows in this diagram except for possibly the top one are conservative, hence so is the top one, as desired. □

Proposition 19. *Let $f\colon X \to Y$ be a morphism of ringed spaces. Let $V \subseteq Y$ be open, and let $U = f^{-1}V$. Let $j\colon V \to Y$, $j'\colon U \to X$ be the inclusion morphisms, and let $f' = f|_U$. Then $j^* Rf_* = Rf'_*(j')^*$.*

Proof. Use the identity $j^* f_* = f'_*(j')^*$, and the fact that j^* is exact, and has an exact left adjoint $j_!$, so it preserves right K-injective resolutions. □

In the same way, we see the following.

Proposition 20. *Let X be a ringed space, let $U \subseteq X$ be open, and let $j\colon U \to X$ be the inclusion morphism. Then $j^* R\,\mathcal{H}om_{\mathcal{O}_X}(-,-) = R\,\mathcal{H}om_{\mathcal{O}_X}(j^*-, j^*-)$.*

3.2 $f^!$

As an application of the above, we show the following.

Proposition 21. *Let $f\colon X \to Y$ be a morphism of qcqs schemes. Then $Rf_*\colon D(\mathcal{Q}_X) \to D(\mathcal{Q}_Y)$ respects coproducts.*

Proof. We need to show that for any collection A_λ of objects in $D(\mathcal{Q}_X)$, the natural map

$$\bigoplus Rf_* A_\lambda \longrightarrow Rf_*\left(\bigoplus A_\lambda\right)$$

is an isomorphism. By the previous, we may check this locally, so we assume that $Y = \operatorname{Spec} R$ is affine.

We proceed by qcqs induction. First note that for $U \subseteq X$ open affine, say $U = \operatorname{Spec} S$, then the category of quasi-coherent $\mathcal{O}_U$-modules is equivalent to that of S-modules, and f_* sends an S-module to its restriction as an R-module. This respects coproducts.

Now suppose that for $U, V \subseteq X$ are quasi-compact open subsets such that our claim is true for U, V, and $U \cap V$. Then, for all objects A of $D(\mathcal{Q}_{U\cup V})$, the Mayer-Vietoris sequence

$$0 \longrightarrow (f|_{U\cup V})_* A \longrightarrow (f|_U)_* A|_U \oplus (f|_V)_* A|_V \longrightarrow (f|_{U\cap V})_* A|_{U\cap V} \longrightarrow 0$$

induces a Mayer-Vietoris triangle $T(A)$

$$R(f|_{U\cup V})_* A \longrightarrow R(f|_U)_* A|_U \oplus R(f|_V)_* A|_V \longrightarrow R(f|_{U\cap V})_* A|_{U\cap V} \longrightarrow R(f|_{U\cup V})_* A[1].$$

Moreover, we have, for any collection A_λ of objects of $D(\mathcal{Q}_{U\cup V})$, a morphism of triangles

$$\bigoplus T(A_\lambda) \longrightarrow T\left(\bigoplus A_\lambda\right)$$

of which two of the arrows are isomorphisms by assumption. Hence the third,

$$\bigoplus Rf_* A_\lambda \longrightarrow Rf_*\left(\bigoplus A_\lambda\right)$$

is an isomorphism as well, as desired. □

Corollary 22. *Let $f\colon X \to Y$ be a morphism of qcqs schemes. Then $Rf_*\colon D(\mathcal{Q}_X) \to D(\mathcal{Q}_Y)$ has a right adjoint $f^!\colon D(\mathcal{Q}_Y) \to D(\mathcal{Q}_X)$.*

Under some additional assumptions, this right adjoint $f^!$ also commutes with coproducts. The key results, which we will use as a black box, are the following.

Definition 23. Let X be a ringed space. A *strictly perfect complex* is a bounded complex of finite free $\mathcal{O}_X$-modules. A *perfect complex* is a complex $A \in K(\mathcal{O}_X)$ locally quasi-isomorphic to a strictly perfect complex.

Theorem 24. *Let X be a qcqs scheme. Then the compact objects of $D(\mathcal{Q}_X)$ are precisely those quasi-isomorphic to perfect complexes.*

Theorem 25 (Lipman and Neeman [2, Ex. 2.2])**.** *Let $f\colon X \to Y$ be a flat proper morphism of qcqs schemes, that is locally of finite presentation. Then $Rf_*\colon D(\mathcal{Q}_X) \to D(\mathcal{Q}_Y)$ sends perfect complexes to perfect complexes.*

The fact that $f^!$ commutes with coproducts in the situation above now follows from the following general statement.

Theorem 26. *Let $\mathcal{S}$ be a compactly generated triangulated category, and let T be any triangulated category. Let $F\colon \mathcal{S} \to \mathcal{T}$ be a triangulated functor respecting coproducts, and let G be its right adjoint. Let S be a generating set for $\mathcal{S}$ consisting of compact elements. Then G respects coproducts if and only if for all $s \in S$, $F(s)$ is a compact object of $\mathcal{T}$.*

Proof. First suppose G respects coproducts, and let $s \in S$. Then, for any collection A_λ of objects of $\mathcal{T}$,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{T}}\Big(F(s), \bigoplus A_\lambda\Big) &= \operatorname{Hom}_{\mathcal{S}}\Big(s, G\Big(\bigoplus A_\lambda\Big)\Big)\\ &= \operatorname{Hom}_{\mathcal{S}}\Big(s, \bigoplus G(A_\lambda)\Big)\\ &= \bigoplus \operatorname{Hom}_{\mathcal{S}}(s, G(A_\lambda))\\ &= \bigoplus \operatorname{Hom}_{\mathcal{T}}(F(s), A_\lambda),\end{aligned}$$

hence $F(s)$ is compact.

Conversely, suppose that $F(s)$ is compact for all $s \in S$. Moreover, let A_λ be a collection of objects in $\mathcal{T}$. Then for all $s \in S$, we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{S}}\Big(s, G(\bigoplus A_\lambda)\Big) &= \operatorname{Hom}_{\mathcal{T}}\Big(F(s), \bigoplus A_\lambda\Big)\\ &= \bigoplus \operatorname{Hom}_{\mathcal{T}}(F(s), A_\lambda)\\ &= \bigoplus \operatorname{Hom}_{\mathcal{S}}(s, G(A_\lambda))\\ &= \operatorname{Hom}_{\mathcal{S}}\Big(s, \bigoplus G(A_\lambda)\Big),\end{aligned}$$

i.e. the natural transformation

$$\varphi\colon \operatorname{Hom}_{\mathcal{S}}(-, \oplus G(A_\lambda)) \longrightarrow \operatorname{Hom}_{\mathcal{S}}\Big(-, G\big(\oplus A_\lambda\big)\Big)$$

is such that $\varphi(s)$ is an isomorphism for all $s \in S$.

Considering the distinguished triangle

$$\oplus G(A_\lambda) \longrightarrow G(\oplus A_\lambda) \longrightarrow Z \longrightarrow \oplus G(A_\lambda)[1],$$

we see that $\mathrm{Hom}_{\mathcal{S}}(s, Z) = 0$ for all $s \in S$. Since S generates $\mathcal{S}$, it follows that $Z = 0$, hence $\bigoplus G(A_\lambda) \to G(\bigoplus A_\lambda)$ is an isomorphism. $\square$

Corollary 27. *Let $f\colon X \to Y$ be a flat proper morphism of qcqs schemes, that is locally of finite presentation. Then $f^!$ respects coproducts.*

3.3 Projection formula

Let $f\colon X \to Y$ be a morphism of schemes. Let A be an object of $D(\mathcal{O}_X)$, and let B be an object of $D(\mathcal{O}_Y)$. Consider the counit $Lf^*Rf_*A \to A$ of the adjoint pair (Lf^*, Rf_*) of functors. This gives a morphism

$$Lf^*(B \overset{L}{\otimes}_{\mathcal{O}_Y} Rf_*A) = Lf^*B \overset{L}{\otimes}_{\mathcal{O}_X} Lf^*Rf_*A \longrightarrow Lf^*B \overset{L}{\otimes}_{\mathcal{O}_X} A,$$

hence by adjunction a morphism

$$B \overset{L}{\otimes}_{\mathcal{O}_Y} Rf_*A \longrightarrow Rf_*(Lf^*B \overset{L}{\otimes}_{\mathcal{O}_X} A),$$

functorial in both A and B.

Proposition 28 (Projection formula)**.** *Let $f\colon X \to Y$ be a morphism of qcqs schemes. Let A be an object of $D(\mathcal{Q}_X)$, and let B be an object of $D(\mathcal{Q}_Y)$. Then the natural morphism*

$$B \overset{L}{\otimes}_{\mathcal{O}_Y} Rf_*A \longrightarrow Rf_*(Lf^*B \overset{L}{\otimes}_{\mathcal{O}_X} A)$$

is an isomorphism.

Proof. We check this locally on Y, so assume that Y is affine, so that X is a qcqs scheme. For a fixed object A of $D(\mathcal{Q}_X)$, we obtain a natural transformation

$$\varphi_A\colon - \overset{L}{\otimes}_{\mathcal{O}_Y} Rf_*A \longrightarrow Rf_*\big(Lf^*(-) \overset{L}{\otimes}_{\mathcal{O}_X} A\big)$$

of functors $D(\mathcal{Q}_Y) \to D(\mathcal{Q}_Y)$. As X is qcqs, Rf_* respects coproducts, and as derived tensor products and Lf^* are left adjoints, they respect coproducts as well. Hence both functors respect coproducts.

Let $\mathcal{R}$ be the full subcategory of $D(\mathcal{Q}_Y)$ consisting of all $B[n]$ such that $\varphi_A\big(B[n]\big)$ is an isomorphism. By the above, we know that $\mathcal{R}$ is closed under coproducts.

We show that $\mathcal{O}_Y$ is an object of $\mathcal{R}$. Note that $\mathcal{O}_Y[n]$ is K-flat in $D(\mathcal{Q}_Y)$, hence $Lf^*\mathcal{O}_Y[n] = f^*\mathcal{O}_Y[n] = \mathcal{O}_X[n]$. The complex $\mathcal{O}_X[n]$ is K-flat in $D(\mathcal{Q}_X)$, so we deduce that $\mathcal{O}_Y[n] \overset{L}{\otimes}_{\mathcal{O}_Y} Rf_*A = Rf_*A[n]$ and $Rf_*\big(\mathcal{O}_X[n] \overset{L}{\otimes}_{\mathcal{O}_X} A\big) = Rf_*A[n]$. By construction, the morphism

$$Lf^*Rf_*A[n] = Lf^*\big(\mathcal{O}_Y[n] \overset{L}{\otimes}_{\mathcal{O}_Y} Rf_*A\big) \longrightarrow Lf^*\mathcal{O}_Y[n] \overset{L}{\otimes}_{\mathcal{O}_X} A = A[n]$$

induced by adjunction, is the counit of adjunction, hence our original morphism $Rf_*A[n] \to Rf_*A[n]$ is the identity.

Now note that $Rf_*, Lf^*, \overset{L}{\otimes}_{\mathcal{O}_X}, \overset{L}{\otimes}_{\mathcal{O}_Y}$ are triangulated functors, so both of our functors are triangulated as well, hence $\mathcal{R}$ is closed under triangles. Since $\{\mathcal{O}_Y[n]\}$ generates $D(\mathcal{Q}_Y)$, it hence follows that $\mathcal{R} = D(\mathcal{Q}_Y)$, as desired. $\square$

3.4 The dualising complex

Let $f\colon X \to Y$ be a morphism of qcqs schemes. Let B and C be objects of $D(\mathcal{Q}_Y)$. Consider the counit $Rf_* f^! C \to C$ of the adjoint pair $(Rf_*, f^!)$ of functors. This gives by the projection formula, a morphism

$$Rf_*(Lf^* B \overset{L}{\otimes}_{\mathcal{O}_X} f^! C) \;=\!=\!=\; B \overset{L}{\otimes}_{\mathcal{O}_Y} Rf_* f^! C \longrightarrow B \overset{L}{\otimes}_{\mathcal{O}_Y} C,$$

and by adjunction, we get a morphism

$$Lf^* B \overset{L}{\otimes}_{\mathcal{O}_X} f^! C \longrightarrow f^!(B \overset{L}{\otimes}_{\mathcal{O}_Y} C),$$

functorial in both B and C.

Theorem 29. *Let $f\colon X \to Y$ be a flat proper morphism of qcqs schemes, that is locally of finite presentation. Then for all objects B, C of $D(\mathcal{Q}_Y)$, the natural map*

$$\varphi\colon Lf^* B \overset{L}{\otimes}_{\mathcal{O}_X} f^! C \longrightarrow f^!(B \overset{L}{\otimes}_{\mathcal{O}_Y} C)$$

is an isomorphism.

Proof. Let B be a perfect complex in $D(\mathcal{Q}_Y)$. Let $B^\vee = R\mathcal{H}om_{\mathcal{O}_Y}(B, \mathcal{O}_Y)$ be its dual. By [SP08DQ], it is perfect as well, $(B^\vee)^\vee = B$, and $R\mathcal{H}om_{\mathcal{O}_Y}(B, D) = B^\vee \overset{L}{\otimes}_{\mathcal{O}_Y} D$ for all objects D of $D(\mathcal{Q}_Y)$. Then $Lf^*(B^\vee) = (Lf^* B)^\vee$, by applying [SP08DM] locally. We show that the natural map $Lf^* B \overset{L}{\otimes}_{\mathcal{O}_X} f^! C \to f^!(B \overset{L}{\otimes}_{\mathcal{O}_Y} C)$ is an isomorphism. For all objects A of $D(\mathcal{Q}_X)$, we have (using that B and $Lf^* B$ are perfect complexes)

$$\begin{aligned}
\operatorname{Hom}_{D(\mathcal{Q}_X)}(A, Lf^* B \overset{L}{\otimes}_{\mathcal{O}_X} f^! C) &= \operatorname{Hom}_{D(\mathcal{Q}_X)}\big(A, R\mathcal{H}om_{\mathcal{O}_X}(Lf^* B^\vee, f^! C)\big) \\
&= \operatorname{Hom}_{D(\mathcal{Q}_X)}(A \overset{L}{\otimes}_{\mathcal{O}_X} Lf^* B^\vee, f^! C) \\
&= \operatorname{Hom}_{D(\mathcal{Q}_Y)}\big(Rf_*(A \overset{L}{\otimes}_{\mathcal{O}_X} Lf^* B^\vee), C\big) \\
&= \operatorname{Hom}_{D(\mathcal{Q}_Y)}(Rf_* A \overset{L}{\otimes}_{\mathcal{O}_Y} B^\vee, C) \\
&= \operatorname{Hom}_{D(\mathcal{Q}_Y)}(Rf_* A, B \overset{L}{\otimes}_{\mathcal{O}_Y} C) \\
&= \operatorname{Hom}_{D(\mathcal{Q}_X)}\big(A, f^!(B \overset{L}{\otimes}_{\mathcal{O}_Y} C)\big)
\end{aligned}$$

and by construction, this identification is given by the given natural map.

Now the functors $f^!(- \overset{L}{\otimes}_{\mathcal{O}_Y} C)$ and $Lf^* - \overset{L}{\otimes}_{\mathcal{O}_X} f^! C$ are triangulated functors respecting coproducts. Hence the full subcategory of $D(\mathcal{Q}_Y)$ of objects B such that the natural map φ is an isomorphism is triangulated, contains all the compact objects, and is closed under coproducts, so it must be $D(\mathcal{Q}_Y)$ itself, as desired. □

This motivates the following definition.

Definition 30. Let $f\colon X \to Y$ be a flat proper morphism of qcqs schemes that is locally of finite presentation. Then the *dualising complex* is $f^! \mathcal{O}_Y$.

4 Grothendieck-Serre duality

First let $f\colon X \to Y$ be a morphism of ringed spaces, and let A, B be objects of $D(\mathcal{O}_X)$. Let $A \to I$, $B \to J$ be K-injective resolutions. Then we have a natural map

$$Rf_* R\,\mathcal{H}om_{\mathcal{O}_X}(A, B) = f_*\,\mathcal{H}om_{\mathcal{O}_X}(I, J) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(f_* I, f_* J) = \mathcal{H}om_{\mathcal{O}_Y}(Rf_* A, Rf_* B).$$

Now suppose that $f\colon X \to Y$ is a flat proper morphism of qcqs schemes that is locally of finite presentation. Let A be an object of $D(\mathcal{Q}_X)$ and B be an object of $D(\mathcal{Q}_Y)$. Then using the counit $Rf_* f^! B \to B$, we get a natural map

$$Rf_* R\,\mathcal{H}om_{\mathcal{O}_X}(A, f^! B) \longrightarrow R\,\mathcal{H}om_{\mathcal{O}_Y}(Rf_* A, Rf_* f^! B) \longrightarrow R\,\mathcal{H}om_{\mathcal{O}_Y}(Rf_* A, B).$$

Theorem 31. *Let $f\colon X \to Y$ be a flat proper morphism of qcqs schemes that is locally of finite presentation. Let A be an object of $D(\mathcal{Q}_X)$ and let B be an object of $D(\mathcal{Q}_Y)$. Then the natural map*

$$Rf_* R\,\mathcal{H}om_{\mathcal{O}_X}(A, f^! B) \longrightarrow R\,\mathcal{H}om_{\mathcal{O}_Y}(Rf_* A, B)$$

is an isomorphism.

Before we prove this theorem, we define the following.

Definition 32. Let X be a ringed space, and let $\mathcal{E}$ be an $\mathcal{O}_X$-module. Let $Z \subseteq X$ be a closed subset, let $U = X - Z$, and let $j\colon U \to X$ be the inclusion. Then the $\mathcal{O}_X$-module $\underline{\Gamma}_Z(\mathcal{E})$ is the kernel of the unit $\mathcal{E} \to j_* j^* \mathcal{E}$ of adjunction.

By the snake lemma, $\underline{\Gamma}_Z$ is a left exact functor, hence we can compute its right derived functor $R\underline{\Gamma}_Z$ (also called the *local cohomology functor*) using K-injective resolutions. As for every injective $\mathcal{O}_X$-module $\mathcal{E}$, the sequence

$$0 \longrightarrow \underline{\Gamma}_Z \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow j_* j^* \mathcal{E} \longrightarrow 0$$

is exact, and any complex in $K(\mathcal{O}_X)$ admits a K-injective resolution with injective terms, we obtain a distinguished triangle

$$R\underline{\Gamma}_Z A \longrightarrow A \longrightarrow Rj_* j^* A \longrightarrow R\underline{\Gamma}_Z A[1]$$

in $D(\mathcal{O}_X)$ which is functorial in A.

We use the local cohomology functor to show the following.

Lemma 33. *Let $f\colon X \to Y$ be a flat proper morphism of qcqs schemes that is locally of finite presentation. Let $V \subseteq Y$ be a quasi-compact open subset, let $U = f^{-1}V$, let $j\colon V \to Y$ and $j'\colon U \to X$ be the corresponding open immersions, and let $f' = f|_U$. Then $(j')^* f^! = (f')^! j^*$.*

Proof. First note that $j^* Rf_* = Rf'_*(j')^*$. So taking right adjoints, we find that $f^! Rj_* = Rj'_*(f')^!$, and hence that $(j')^* f^! Rj_* j^* = (j')^* Rj'_*(f')^! j^*$. As $(j')^* j'_*$ is the identity functor on $K(\mathcal{O}_U)$, it follows that $(j')^* Rj'_*$ is the identity functor on $D(\mathcal{Q}_U)$. We deduce that $(j')^* f^! Rj_* j^* = (f')^! j^*$.

It now suffices to show that, functorially in A (in $D(\mathcal{Q}_Y)$), the unit $A \to Rj_* j^* A$ induces an isomorphism $(j')^* f^! A \to (j')^* f^! Rj_* j^*$ in $D(\mathcal{Q}_Y)$. It suffices to show that $(j')^* f^! R\underline{\Gamma}_Z = 0$. Let B be an object of $D(\mathcal{Q}_Y)$. Then $R\underline{\Gamma}_Z B$ has cohomology supported on Z by definition. Hence $f^! R\underline{\Gamma}_Z B = Lf^* R\underline{\Gamma}_Z B \overset{L}{\otimes}_{\mathcal{O}_X} f^! \mathcal{O}_Y$ has cohomology supported on $f^{-1}Z$, i.e. it becomes zero after applying $(j')^*$ to it. □

Proof of Theorem 31. By the previous lemma, after applying $\mathrm{H}^0(V, -)$ for all quasi-compact open subsets $V \subseteq Y$, we obtain an isomorphism. Hence it induces isomorphisms on cohomology, as desired. $\square$

We have a variant for bounded below complexes as well, although its proof is much harder due to lack of derived pullbacks, so we will only state this version here.

Theorem 34. *Let $f\colon X \to Y$ be a flat proper morphism of qcqs schemes that is locally of finite presentation. Let A be an object of $D^+(\mathcal{Q}_X)$ and let B be an object of $D^+(\mathcal{Q}_Y)$. Then $Rf_*\colon D^+(\mathcal{Q}_X) \to D^+(\mathcal{Q}_Y)$ has a right adjoint $f^!$ and the natural map*

$$Rf_* R\,\mathcal{H}om_{\mathcal{O}_X}(A, f^! B) \longrightarrow R\,\mathcal{H}om_{\mathcal{O}_Y}(Rf_* A, B)$$

is an isomorphism.

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