Cohomology on the étale sites

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Note: Most (if not all) of this can be found in one form or another in the Stacks Project. Explicit references are given rarely, but if they are given, they are of the form [SP,xxxx] referring to tag xxxx in the Stacks Project.

1 Introduction

If X is a smooth projective variety over $\mathbb{C}$, then, for the archimedean topology on $X(\mathbb{C})$, we have $\mathrm{H}^{2\dim X}_{sing}\big(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}\big) = \mathbb{Z}/n\mathbb{Z}$ for non-negative n. We would like to have an algebraic theory of cohomology with similar kinds of results. Usual sheaf cohomology doesn't suffice here, since $\mathrm{H}^i(X, \mathbb{Z}/n\mathbb{Z})$ (or $\mathrm{H}^i(X, \underline{A}_X)$ for any abelian group A) vanishes for all irreducible schemes X and all $i > 0$; as the restriction maps of a constant sheaf on an irreducible space are all equal to the identity map, all constant sheaves on irreducible spaces are flasque, hence all higher cohomology vanishes.

This is solved by "adding more open subsets to X". We will now proceed with explaining exactly what we mean by the previous sentence.

2 Étale morphisms

We first define (or rather, recall the definition of) our "open subsets".

Definition 2.1. Let $f\colon X \to S$ be a morphism of schemes. We say that f is *formally étale* if for all rings A, morphisms $\operatorname{Spec} A \to S$, and ideals I of A with $I^2 = 0$, the canonical map $X_S(A) \to X_S(A/I)$ is bijective. Moreover, we say that f is *étale* if f is formally étale and locally of finite presentation.

Remark 2.2. Note that the property of a morphism being locally of finite presentation can also be expressed purely in terms of the functor of points; a morphism $f\colon X \to S$ is locally of finite presentation if and only if the functor X_S commutes with filtered colimits, i.e. for all filtered diagrams of rings A_i together with morphisms $\operatorname{Spec} A_i \to S$, we have $X_S(\operatorname{colim}_i A_i) = \operatorname{colim}_i X_S(A_i)$. So étaleness of a morphism of schemes can be expressed purely in terms of its functor of points.

Étale morphisms to the spectrum of a field have a particularly easy characterisation.

Proposition 2.3. *Let k be a field. Then $S \to \operatorname{Spec} k$ is étale if and only if S is the disjoint union of $\operatorname{Spec} k_i$ with k_i a finite and separable extension of k.*

This is particularly useful in combination with the following characterisation.

Proposition 2.4. *Let $f\colon X \to S$ be a morphism of schemes. Then f is étale if and only if it is flat, locally of finite presentation, and all of its fibres are étale.*

For affine schemes, we have a characterisation in more concrete terms.

Proposition 2.5. *Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of schemes. Then f is étale if and only if B is as an A-algebra of the form*

$$B = A[x_1, \dots, x_s]/(f_1, \dots, f_s)$$

with $|\partial f_j / \partial x_i|_{i,j} \in B^\times$.

Finally, we give some properties of étale morphisms.

Proposition 2.6.

- *The composition of étale morphisms is étale.*
- *The base change of an étale morphism is étale.*
- *Any morphism between étale S-schemes is étale, for any scheme S.*
- *Being étale is local on the source.*
- *Being étale is local on the base.*

3 Étale sites

The following notion, the notion of a *site*, is a generalisation of the notion of a topology.

Definition 3.1. A *site* is a pair $(\mathcal{C}, \operatorname{Cov}(\mathcal{C}))$ where

- $\mathcal{C}$ is a category;
- $\operatorname{Cov}(\mathcal{C})$ is a collection of families of morphisms $\{\varphi_i\colon U_i \to U\}$ in $\mathcal{C}$, called *coverings*, such that
  - if $\varphi\colon V \to U$ is an isomorphism, then $\{\varphi\} \in \operatorname{Cov}(\mathcal{C})$;
  - if $\{\varphi_i\colon U_i \to U\}$ in $\operatorname{Cov}(\mathcal{C})$, and for all i, $\{\psi_{ij}\colon U_{ij} \to U_i\}$ in $\operatorname{Cov}(\mathcal{C})$, then
  $$\{\varphi_i \psi_{ij}\colon U_{ij} \to U\} \in \operatorname{Cov}(\mathcal{C});$$
  - if $\{\varphi_i\colon U_i \to U\}$ in $\operatorname{Cov}(\mathcal{C})$, then for all morphisms $V \to U$ in $\mathcal{C}$
    * the fibre product $V_i = V \times_U U_i$ exists in $\mathcal{C}$ for all i;
    * the set $\{\varphi_{i,V}\colon V_i \to V\} \in \operatorname{Cov}(\mathcal{C})$.

Let us give some examples.

Example 3.2. Let S be a scheme. Then the category $\operatorname{Op}(S)$ of open subschemes of S, where $\{\varphi_i\colon U_i \to U\}$ is a covering if and only if the φ_i are jointly surjective, is a site. We call this the *small Zariski site* S_{Zar} of S.

Next, we give the main examples of sites we consider in this seminar.

Example 3.3. Let S be a scheme. Then the category Sch_S, where $\{\varphi_i\colon U_i \to U\}$ is a covering if and only if the φ_i are étale and jointly surjective, is a site, the *big étale site* $\operatorname{Sch}_{S,\text{ét}}$ of S.

Example 3.4. Let S be a scheme. Then the full subcategory of Sch_S of étale S-schemes, where $\{\varphi_i\colon U_i \to U\}$ is a covering if and only if the φ_i are jointly surjective, is a site, the *small étale site* $S_{\text{ét}}$ of S.

Here is an indication why the étale topology on a scheme is the "right" topology to consider: Note that it is not true that any smooth S-scheme X (Zariski-)locally admits an open immersion into some $\mathbb{A}^d_S$, the following slightly weaker statement is true.

Proposition 3.5 ([SP,054L])**.** *Let X be an S-scheme. Then X is smooth over S if and only if there exists a (Zariski) cover $\{X_i \to X\}$ of X such that X_i admits an étale S-morphism into some $\mathbb{A}^d_S$.*

So in a sense, smooth S-schemes are precisely those that can be obtained by gluing "open subsets" of $\mathbb{A}^d_S$, with respect to the étale topology on $\mathbb{A}^d_S$, and this fact should remind you of a definition of a smooth manifold.

4 Sheaves on sites

We now define the notion of a sheaf on a site. Do check as a (rather easy) exercise that on the site S_{Zar}, we get the familiar notion back.

Definition 4.1. Let $\mathcal{C}$ be a site. A *presheaf* on $\mathcal{C}$ is a functor $\mathcal{F}\colon \mathcal{C}^{\text{op}} \to \text{Set}$. A presheaf $\mathcal{F}$ on $\mathcal{C}$ is *separated* if for all coverings $\{\varphi_i\colon U_i \to U\}$ the map

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i)$$

is injective. A presheaf $\mathcal{F}$ on $\mathcal{C}$ is a *sheaf* if for all coverings $\{\varphi_i\colon U_i \to U\}$ the diagram

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

where the left map is given by $x \mapsto (x|_{U_i})$ and the two maps on the right are given by $(x_i) \mapsto (x_i|_{U_i\times_U U_j})$ and $(x_i) \mapsto (x_j|_{U_i\times_U U_j})$, is an equaliser diagram. A *morphism* of (pre-)sheaves is a morphism of functors.

From this definition, one can also deduce what a sheaf of abelian groups, rings, modules, etc. is in this setting. Denote the category of sheaves on $\mathcal{C}$ by $\text{Sh}(\mathcal{C})$, and the category of presheaves on $\mathcal{C}$ by $\text{PSh}(\mathcal{C})$.

Now, as in the familiar case, one would like to sheafify presheaves, i.e. give a left adjoint to the forgetful functor $\text{Sh}(\mathcal{C}) \to \text{PSh}(\mathcal{C})$.

Proposition 4.2. *Let $\mathcal{C}$ be a site. Then the forgetful functor* $\text{Sh}(\mathcal{C}) \to \text{PSh}(\mathcal{C})$ *admits a left adjoint.*

We will only give the construction of the sheafification, and even then, only modulo a lot of details.

Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$, and let $\mathcal{U} = \{U_i \to U\}$ be a covering. Define the *zeroth Čech cohomology*

$$\check{\text{H}}^0(\mathcal{U},\mathcal{F}) = \Big\{(s_i) \in \prod_i \mathcal{F}(U_i) : (\forall i,j)(s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j})\Big\}.$$

Of course, we have a canonical map $\mathcal{F}(U) \to \check{\text{H}}^0(\mathcal{U},\mathcal{F})$, which is a bijection for all coverings $\mathcal{U}$ if and only if $\mathcal{F}$ is a sheaf. So let's try, for all objects U of $\mathcal{C}$, replacing $\mathcal{F}(U)$ by $\text{colim}_{\mathcal{U}}\, \check{\text{H}}^0(\mathcal{U},\mathcal{F})$, where $\mathcal{U}$ runs through all coverings of U (taking as morphisms the refinements).

We first need to make sense of the latter expression. Let $\mathcal{U} = \{\varphi_i\colon U_i \to U : i \in I\}$, $\mathcal{V} = \{\psi_j\colon V_j \to V : j \in J\}$. Then a *morphism* $\mathcal{V} \to \mathcal{U}$ of coverings is given by a morphism $\chi\colon V \to U$, a map $\alpha\colon J \to I$, and morphisms $\chi_j\colon V_j \to U_{\alpha(j)}$ for all j, such that for all j, we have $\varphi_{\alpha(j)}\chi_j = \chi\psi_j$. Moreover, a *refinement* is a morphism $\mathcal{V} \to \mathcal{U}$ of coverings with $V = U$ and $\chi = \text{id}_U$.

Now a morphism $\chi\colon \mathcal{V} \to \mathcal{U}$ defines a map $\chi^*\colon \check{\text{H}}^0(\mathcal{U},\mathcal{F}) \to \check{\text{H}}^0(\mathcal{V},\mathcal{F})$, $(s_i) \mapsto (\chi_j^* s_{\alpha(j)})$. We (well, by that we mean the reader) have to check the following.

- The map χ^* is well-defined.
- The map χ^* is independent of α and the χ_j for all j.

Define $\mathcal{F}^+(U) = \text{colim}_{\mathcal{U}}\, \check{\text{H}}^0(\mathcal{U},\mathcal{F})$, where $\mathcal{U}$ runs through all coverings of U (where the morphisms are the refinements). Note that this colimit actually is directed, despite its index category not being codirected; any two refinements $\mathcal{U}' \to \mathcal{U}$ of coverings of U define the same morphism $\check{\text{H}}^0(\mathcal{U},\mathcal{F}) \to \check{\text{H}}^0(\mathcal{U}',\mathcal{F})$, and for any two coverings $\mathcal{U}$ and $\mathcal{U}'$ of U, there exists a third covering $\mathcal{U}''$ of U, and refinements $\mathcal{U}'' \to \mathcal{U}$ and $\mathcal{U}'' \to \mathcal{U}'$. Therefore an element of $\mathcal{F}^+(U)$ is just an equivalence class of pairs $(s,\mathcal{U})$ with $\mathcal{U}$ a covering of U and $s \in \check{\text{H}}^0(\mathcal{U},\mathcal{F})$, where two pairs $(s,\mathcal{U})$ and $(t,\mathcal{V})$ are equivalent if and only if there exists a covering $\mathcal{W}$ and refinements $\mathcal{W} \to \mathcal{U}$ and $\mathcal{W} \to \mathcal{V}$ such that the images of s and t in $\check{\mathrm{H}}^0(\mathcal{W},\mathcal{F})$ are the same.

Theorem 4.3 ([SP,00WB])**.** *Let $\mathcal{F}$ be a presheaf on a site $\mathcal{C}$.*

- *$\mathcal{F}^+$ is a presheaf.*
- *There is a canonical map $\mathcal{F} \to \mathcal{F}^+$.*
- *$\mathcal{F}^+$ is separated.*
- *If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf, and the canonical map $\mathcal{F} \to \mathcal{F}^+$ is injective.*
- *$\mathcal{F}^\# = (\mathcal{F}^+)^+$ is the sheafification of $\mathcal{F}$, and therefore $-^\#$ is a left adjoint of the forgetful functor.*

We next give some examples of (pre-)sheaves on the étale sites of a scheme. Note first that any (pre-)sheaf on the big étale site of a scheme, by restriction gives rise to a (pre-)sheaf on the corresponding small étale site. Hence we will only give examples of (pre-)sheaves on the big étale site.

Example 4.4. Let S be a scheme.

- The presheaf $\mathcal{O}_S$ (or $\mathbb{G}_{a,S}$) of rings on $\mathrm{Sch}_{S,\text{ét}}$ is defined by $T \mapsto \mathcal{O}_T(T)$.
- The presheaf $\mathcal{O}_S^\times$ (or $\mathbb{G}_{m,S}$) of groups on $\mathrm{Sch}_{S,\text{ét}}$ is defined by $T \mapsto \mathcal{O}_T(T)^\times$.
- The sheaf $\underline{\mathbb{Z}/n\mathbb{Z}}_S$ on $\mathrm{Sch}_{S,\text{ét}}$ is the sheafification of the constant presheaf with values in $\mathbb{Z}/n\mathbb{Z}$.
- The constant presheaf on $\mathrm{Sch}_{S,\text{ét}}$ with values in A is a sheaf if and only if $A = 0$.

In the next lecture, we will see that $\mathcal{O}_S$ and $\mathcal{O}_S^\times$ are in fact sheaves, and that for all S-schemes U, the group $\underline{\mathbb{Z}/n\mathbb{Z}}_S(U)$ is the group of locally constant maps $U \to \mathbb{Z}/n\mathbb{Z}$.

5 Cohomology on a site

The following facts allow us to define cohomology on sites in the usual way, as the right derived functors of the sections functors.

Theorem 5.1. *Let $\mathcal{C}$ be a site. Then the category $\mathrm{Ab}(\mathcal{C})$ is abelian and has enough injectives. Moreover, the sections functor $\Gamma(U,-)\colon \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}$ is left exact for all objects U of $\mathcal{C}$.*

Definition 5.2. Let $\mathcal{C}$ be a site, U be an object of $\mathcal{C}$, and let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then the *n-th cohomology group of $\mathcal{F}$* is

$$\mathrm{H}^n(U,\mathcal{F}) = R^n\Gamma(U,\mathcal{F}) = \mathrm{H}^n\big(\mathcal{I}^\cdot(U)\big),$$

for $\mathcal{F} \to \mathcal{I}^\cdot$ an injective resolution of $\mathcal{F}$.

In the special case that $\mathcal{C}$ is either $\mathrm{Sch}_{S,\text{ét}}$ or $S_{\text{ét}}$ for a scheme S, we denote these cohomology groups by $\mathrm{H}^n_{\text{ét}}(U,\mathcal{F})$. The following theorem says that there is no ambiguity in doing so.

Theorem 5.3. *Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathrm{Sch}_{S,\acute{e}t}$. Then for all U/S étale, we have $\mathrm{H}^n_{\acute{e}t}(U,\mathcal{F}) = \mathrm{H}^n_{\acute{e}t}(U,\mathcal{F}|_{S_{\acute{e}t}})$.*

Proof. We first define two functors $c_*\colon \mathrm{Sh}(\mathrm{Sch}_{S,\text{ét}}) \to \mathrm{Sh}(S_{\text{ét}})$ and $c^{-1}\colon \mathrm{Sh}(S_{\text{ét}}) \to \mathrm{Sh}(\mathrm{Sch}_{S,\text{ét}})$. The former one is defined by

$$c_*\mathcal{F} = \mathcal{F}|_{S_{\text{ét}}} = \big(U \mapsto \mathcal{F}(U)\big),$$

and the latter is defined by

$$c^{-1}\mathcal{F} = \big(U \mapsto \mathrm{colim}_V\, \mathcal{F}(V)\big)^\#$$

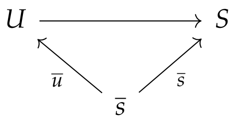
where V runs through the étale S-schemes such that the S-scheme U factors through V. Note that c^{-1} is left adjoint to c_*. As filtered colimits of injective maps are injective, we see that c^{-1} maps injective maps to injective maps. Hence c_* maps injectives to injectives, and we're done. □

6 Stalks of sheaves on the small étale site

For the small étale site, the theory of stalks of sheaves is similar to that of the small Zariski site, provided we pick the right notion of points and neighbourhoods.

Definition 6.1. Let S be a scheme. A *geometric point* $\bar{s}$ is an S-scheme of the from $\operatorname{Spec} k$ with k separably closed. If its image is $s \in S$, we say that $\bar{s}$ *lies over* s. A *morphism* of geometric points $\bar{s} \to \bar{s}'$ is a morphism of S-schemes.

Definition 6.2. Let S be a scheme, and let $\bar{s}$ be a geometric point of S. An *étale neighbourhood* of $\bar{s}$ is an étale morphism of geometrically pointed schemes $(U, \bar{u}) \to (S, \bar{s})$, i.e. a commutative diagram



with $U \to S$ étale.

Lemma 6.3. *Let S be a scheme, and let $\bar{s}$ be a geometric point of S. The category of étale neighbourhoods of $\bar{s}$ is cofiltered.*

Hence we can define stalks in the usual way.

Definition 6.4. Let S be a scheme, and let $\bar{s}$ be a geometric point of S. The *stalk* of a presheaf $\mathcal{F}$ on $S_{\text{ét}}$ is

$$\mathcal{F}_{\bar{s}} = \operatorname{colim}_{(U,\bar{u})} \mathcal{F}(U),$$

where $(U, \bar{u})$ runs through all étale neighbourhoods of $\bar{s}$.

More concretely, an element of the stalk $\mathcal{F}_{\bar{s}}$ is an equivalence class of triples $(x, U, \bar{u})$, where $(U, \bar{u})$ is an étale neighbourhood of $\bar{s}$ and $x \in \mathcal{F}(U)$, and where two such triples $(x, U, \bar{u})$ and $(y, V, \bar{v})$ are equivalent if and only if there exists an étale neighborhood $(W, \bar{w})$ of $\bar{s}$ and morphisms $(W, \bar{w}) \to (U, \bar{u})$ and $(W, \bar{w}) \to (V, \bar{v})$ such that $x|_W = y|_W$.

Given a morphism $\bar{s} \to \bar{s}'$ of geometric points of S, we get a functor F from the category of étale neighbourhoods of $\bar{s}'$ to the category of étale neighbourhoods of $\bar{s}$, hence also, for all presheaves $\mathcal{F}$, a map $\mathcal{F}_{\bar{s}'} \to \mathcal{F}_{\bar{s}}$. As this F is an equivalence of categories, the map on stalks is an isomorphism.

Proposition 6.5. *Let S be a scheme, let $\mathcal{F}$ be a presheaf on $S_{\text{ét}}$, and let $\bar{s}$ be a geometric point of S. Then $\mathcal{F}_{\bar{s}} = (\mathcal{F}^{\#})_{\bar{s}}$.*

Proposition 6.6. *Let S be a scheme. Then the stalk functors* $\operatorname{PSh}(S_{\text{ét}}) \to \operatorname{Set}$, $\operatorname{Sh}(S_{\text{ét}}) \to \operatorname{Set}$, $\operatorname{PAb}(S_{\text{ét}}) \to \operatorname{Ab}$, *and* $\operatorname{Ab}(S_{\text{ét}}) \to \operatorname{Ab}$ *are exact.*

Proposition 6.7. *Let S be a scheme, and let $\mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on S (of sets or of abelian groups). Then $\mathcal{F} \to \mathcal{G}$ is injective (resp. surjective) if and only if it is injective (resp. surjective) on stalks. A sequence of sheaves of abelian groups is exact if and only if it is exact on stalks.*

7 Stalks of the structure sheaf

In this section we study the relationship between the stalks of the structure sheaf $\mathcal{O}_S$ of a scheme S in the usual sense, and the stalks of the structure sheaf on the small étale site on S. We will see that $\mathcal{O}_{S_{\text{ét}},\bar{s}}$ is a so-called *strict henselisation* of $\mathcal{O}_{S,s}$. Let us start defining this.

Informally, a local ring is *henselian* if it satisfies Hensel's lemma. Since there are many versions of Hensel's lemma, we pick one, and take it as our definition.

Definition 7.1. A local ring $(R, \mathfrak{m}, \kappa)$ is *henselian* if for all monic $f \in R[T]$, and all $a_0 \in \kappa$ such that $f(a_0) = 0$ and $f'(a_0) \neq 0$, there exists an $a \in R$ such that $f(a) = 0$ and $a = a_0$ in κ.

Example 7.2. All complete local rings are henselian.

Some characterisations of this notion. More characterisations can be found in [SP,04GG].

Proposition 7.3. *The following are equivalent for a local ring* $(R, \mathfrak{m}, \kappa)$.

- *R is henselian,*
- *for any* $f \in R[T]$ *and any factorisation* $f = g_0 h_0$ *in* $\kappa[T]$ *with* $\gcd(g_0, h_0) = 1$, *there exists a factorisation* $f = gh$ *in* $R[T]$ *with* $g = g_0$ *and* $h = h_0$,
- *any finite R-algebra is isomorphic to a finite product of local rings.*

Definition 7.4. A local ring $(R, \mathfrak{m}, \kappa)$ is *strictly henselian* if it is henselian and κ is separably closed.

Example 7.5. A separably closed field is strictly henselian.

We now define the (strict) henselisation of a local ring. Recall that an *ind-étale* R-algebra is a filtered colimit of étale R-algebras.

Definition 7.6. Let $(R, \mathfrak{m}, \kappa)$ be a local ring. A *henselisation* of R is a final object R^h in the category of pairs $(S, \mathfrak{q})$ of an ind-étale R-algebra S and a prime ideal $\mathfrak{q}$ lying over $\mathfrak{m}$ such that the canonical map $\kappa \to \kappa(\mathfrak{q})$ is an isomorphism, where the morphisms $(S, \mathfrak{q}) \to (S', \mathfrak{q}')$ are the morphisms $\varphi \colon S \to S'$ such that $\varphi^{-1}\mathfrak{q}' = \mathfrak{q}$.

Proposition 7.7. *Let* $(R, \mathfrak{m}, \kappa)$ *be a local ring. Then henselisations exist. Moreover, if* $R \to R^h$ *is a henselisation, then* $\left(R^h, \mathfrak{m}R^h, \kappa(\mathfrak{m}R^h)\right)$ *is a henselian local ring, and the canonical map* $\kappa \to \kappa(\mathfrak{m}R^h)$ *is an isomorphism.*

Sketch of proof. First note that $R^h = \operatorname{colim}_{(S,\mathfrak{q})} S$, where $(S, \mathfrak{q})$ runs through the pairs as above such that S is étale over R (with morphisms as above as well), defines a henselisation of R.

We show that R^h is local with maximal ideal $\mathfrak{m}$. Let $x \in R^h$. By the description above, we can represent x by a triple $(f, S, \mathfrak{q})$ with S étale over R and $f \in S$. By prime avoidance, we may assume that $\mathfrak{q}$ is the only ideal lying over $\mathfrak{m}$. As $S \otimes_R \kappa(\mathfrak{q})$ is étale over κ, it then follows that $\mathfrak{m}S = \mathfrak{q}$.

Now let $x \in R^h - \mathfrak{m}R^h$. Represent x by a triple $(f, S, \mathfrak{q})$ with $\mathfrak{m}S = \mathfrak{q}$. Then $f \notin \mathfrak{m}S = \mathfrak{q}$, so x is also represented by the triple $(f, S_f, \mathfrak{q}S_f)$, which is invertible. Hence x is invertible in R^h, so R^h is a local ring with maximal ideal $\mathfrak{m}R^h$.

It now remains to show that R^h is henselian. Let P be a monic polynomial in $R^h[T]$, and let a_0 be a zero of P, but not of P'. Then there exists a pair $(S, \mathfrak{q})$ such that P is the image of a monic polynomial Q in $S[T]$. Now consider $S' = S[T]/(Q)$ and $\mathfrak{q}' = (\mathfrak{q}, T - a_0)$. Then $\kappa = \kappa(\mathfrak{q}')$, and there exists a $g \in S' - \mathfrak{q}'$ such that S'_g is étale over S. We obtain a morphism $(S, \mathfrak{q}) \to (S'_g, \mathfrak{q}'S'_g)$, and $T \in S'_g$ defines an element a of R^h such that $P(a) = 0$ in R^h. □

To define strict henselisations, some more care is needed; we want the henselisation of a field to be its separable closure, but separable closures are not unique up to a *unique* isomorphism. We settle this by fixing a separable closure of the residue field beforehand.

Definition 7.8. Let $(R, \mathfrak{m}, \kappa)$ be a local ring, and let κ^{sep} be a separable closure of κ. A *strict henselisation* of R with respect to κ^{sep} is a final object R^{sh} in the category of triples $(S, \mathfrak{q}, \alpha)$ of an ind-étale R-algebra S, a prime ideal $\mathfrak{q}$ lying over $\mathfrak{m}$, and a κ-algebra morphism $\alpha \colon \kappa(\mathfrak{q}) \to \kappa^{\text{sep}}$, where the morphisms $(S, \mathfrak{q}, \alpha) \to (S', \mathfrak{q}', \alpha')$ are the morphisms $\varphi \colon S \to S'$ such that $\varphi^{-1}\mathfrak{q}' = \mathfrak{q}$ and $\alpha = \alpha'\overline{\varphi}$ (where $\overline{\varphi} \colon \kappa(\mathfrak{q}) \to \kappa(\mathfrak{q}')$ is the induced morphism).

The following proposition is proved in a similar way as the previous one.

Proposition 7.9. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring, and let κ^{sep} be a separable closure of κ. Then strict henselisations with respect to κ^{sep} exist. Moreover, if $R \to R^{sh}$ is a strict henselisation, then $\big(R^{sh}, \mathfrak{m}R^{sh}, \kappa(\mathfrak{m}R^{sh})\big)$ is a strictly henselian local ring, and the map $\kappa(\mathfrak{m}R^{sh}) \to \kappa^{\text{sep}}$ is an isomorphism.*

Theorem 7.10. *Let S be a scheme, let $s \in S$, let $\bar{s}$ be a geometric point of S lying over s, and let κ^{sep} be the separable closure of $\kappa = \kappa(s)$ in $\kappa(\bar{s})$. Then the canonical morphism $\mathcal{O}_{S,s} \to \mathcal{O}_{S,\bar{s}}$ from the Zariski local ring to the étale local ring is a strict henselisation of $\mathcal{O}_{S,s}$ with respect to κ^{sep}.*

Sketch of proof. Note that

$$\mathcal{O}_{S,\bar{s}} = \operatorname{colim}_{(U,\bar{u})} \mathcal{O}_U(U),$$

where the colimit runs through the étale neighbourhoods of $\bar{s}$. Taking a (Zariski) affine open neighbourhood $\operatorname{Spec} A$ of s, we get a cofinal system of étale neighbourhoods such that U is affine and factors through $\operatorname{Spec} A$, hence

$$\mathcal{O}_{S,\bar{s}} = \operatorname{colim}_{(B,\mathfrak{q},\alpha)} B = \operatorname{colim}_{(B,\mathfrak{q},\alpha)} B_{\mathfrak{q}}$$

where the colimit is taken over the category of triples $(B, \mathfrak{q}, \alpha)$ with B an étale A-algebra, $\mathfrak{q}$ a prime ideal lying over the prime ideal $\mathfrak{p}$ corresponding to s, and $\alpha \colon \kappa(\mathfrak{q}) \to \kappa^{\text{sep}}$ a κ-algebra morphism. Localising these triples by $\mathfrak{p}$ gives an isomorphism

$$\mathcal{O}_{S,\bar{s}} = \operatorname{colim}_{(B,\mathfrak{q},\alpha)} B,$$

where the colimit is taken over the category of triples with B an étale $A_{\mathfrak{p}}$-algebra. (Note that the collection of local such algebras give a cofinal system.) Hence we are done. □