The pro-étale site, part I

J. Jin

Last compile: 18th November, 2014; 18:57

Note: Most (if not all) of this can be found in one form or another in the Stacks Project. Explicit references are given rarely, but if they are given, they are of the form [SP,xxxx] referring to tag xxxx in the Stacks Project.

1 Introduction

As we have seen in Giulio's talk, étale cohomology of a smooth projective curve X (over a separably closed field with $\mathbb{Z}/n\mathbb{Z}$-coefficients with $n \geq 1$) behaves a lot like singular cohomology, since $\mathrm{H}^0(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$ and $\mathrm{H}^2(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$ are isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and $\mathrm{H}^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$, where g is the genus of the curve X. This no longer holds once we replace $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$, as we will show below.

Definition 1.1. A scheme X is *geometrically unibranch* at $x \in X$ if for all geometric points $\overline{x}$ over x, the scheme $\operatorname{Spec} \mathcal{O}_{X,\overline{x}}$ is irreducible. A scheme X is *geometrically unibranch* if it is geometrically unibranch at all $x \in X$.

For example, this is the case when X is a normal k-scheme by [EGAIV,6.15.6].

We have the following useful characterisation.

Lemma 1.2 ([SP,06DK]). *Let X be a scheme, and let $x \in X$. Then X is geometrically unibranch at x if and only if for all étale $U \to X$, all $u \in U$ mapping to $x \in X$, the scheme* $\operatorname{Spec} \mathcal{O}_{U,u}$ *is irreducible.*

Our goal is the following.

Proposition 1.3. *Let X be a geometrically unibranch, irreducible scheme. Then $\mathrm{H}^1(X_{\text{ét}}, \mathbb{Z}) = 0$.*

Note that we do need the condition that X is geometrically unibranch; for example, if X is a nodal cubic curve, then there does exist a connected $\mathbb{Z}$-torsor over X, namely a chain of $\mathbb{P}^1$, each glued to the next in one point.

We will need to introduce some tools in order to prove Proposition 1.3.

1.1 Cohomology, colimits, and higher derived images

The notion of morphisms of abelian sheaves lying over a morphism of schemes, as mentioned by Bas last week, allows us to define a category $\mathrm{Ab}_{\text{ét}}(\mathrm{Sch})$ in which:

- the objects are pairs $(\mathcal{F}, X)$ of an abelian (étale) sheaf $\mathcal{F}$ and the scheme X it lies over;
- the morphisms $(\mathcal{F}, X) \to (\mathcal{G}, Y)$ are pairs (φ, f) of a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ and the morphism $f\colon Y \to X$ it lies over.

This allows us to state [SP,09YQ] in a very concise way.

Theorem 1.4 ([SP,09YQ]). *Let $(\mathcal{F}, X) = \operatorname{colim}_i(\mathcal{F}_i, X_i)$ be a filtered colimit in* $\mathrm{Ab}_{\text{ét}}(\mathrm{Sch})$*, such that each X_i is qcqs, and all transition maps of schemes are affine. Then*

$$\mathrm{H}^p(X_{\text{ét}}, \mathcal{F}) = \operatorname{colim}_i \mathrm{H}^p(X_{i,\text{ét}}, \mathcal{F}_i).$$

If one expands this formulation, one gets back the formulation as in [SP,09YQ].

Remark 1.5. The proof of this theorem boils down to proving that as sites $X_{\text{ét}} = \operatorname{colim}_i X_{i,\text{ét}}$.

We also have the following étale analogue of a well-known fact in usual sheaf cohomology, and the proof is also exactly the same.

Proposition 1.6 ([SP,03Q8])**.** *Let $f\colon Y \to X$ be a morphism of schemes, and let $\mathcal{G}$ be an abelian étale sheaf on Y. Then $R^p f_* \mathcal{G}$ is the sheaf associated to the presheaf*

$$U \mapsto \mathrm{H}^p(Y \times_X U, \mathcal{G}|_{Y \times_X U}).$$

Combining these results then give the following.

Theorem 1.7 ([SP,03Q9])**.** *Let $f\colon Y \to X$ be a morphism of schemes, let $\mathcal{G}$ be an abelian étale sheaf on Y, and let $\overline{x} \in X$ be a geometric point. Then*

$$(R^p f_* \mathcal{G})_{\overline{x}} = \mathrm{H}^p(Y \times_X \operatorname{Spec} \mathcal{O}_{X,\overline{x}}, \pi_Y^{-1} \mathcal{G}).$$

Proof. Better left as exercise, but reference gives a proof as well. □

1.2 Proof of Proposition 1.3

A feature of the étale topology is that cohomology on X is the same as cohomology on its reduced subscheme, see [SP,04DY]. So we may assume that X is integral.

First of all, let η be the generic point of X, and let $i\colon \eta \to X$ denote the inclusion. Then by adjunction we have a natural map $\varphi\colon \underline{\mathbb{Z}}_X \to i_* \underline{\mathbb{Z}}_\eta$ (which is actually a map of sheaves of rings).

We proceed in a number of steps.

Step 1. We show that for any field k, and $\eta = \operatorname{Spec} k$, we have $\mathrm{H}^1(\eta, \underline{\mathbb{Z}}_\eta) = 0$.

Let $\mathcal{G}$ be an element of $\mathrm{H}^1(\operatorname{Spec} k, \mathbb{Z})$, i.e. an étale $\mathbb{Z}$-torsor on $\operatorname{Spec} k$. Then for some finite Galois extension k'/k, the sheaf $\mathcal{G}|_{\operatorname{Spec} k'}$ is the trivial $\mathbb{Z}$-torsor. This implies that we have an action of $\mathrm{Gal}(k'/k)$ on $\mathbb{Z}$ (as $\mathbb{Z}$-torsor), which is the same as a morphism $\mathrm{Gal}(k'/k) \to \mathbb{Z}$ of groups. But the left hand side is torsion, and the right hand side is torsion-free. Therefore this $\mathrm{Gal}(k'/k)$-action on $\mathbb{Z}$ is the trivial one, and $\mathcal{G}$ is the trivial $\mathbb{Z}$-torsor.

Step 2. We show that $K_{\overline{x}} = \mathcal{O}_{X,\eta} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,\overline{x}}$ is a field.

As $\mathcal{O}_{X,\eta}$ is the localisation of $\mathcal{O}_{X,x}$ by the set S of all non-zero elements, we know $K_{\overline{x}} = S^{-1}\mathcal{O}_{X,\overline{x}}$. Let $s \in \mathcal{O}_{X,\overline{x}}$ be a non-zero element. Then there exists an étale map U to X, and a point $u \in U$ mapping to x such that the field of fractions K' of $\mathcal{O}_{U,u}$ is finite separable over the field of fractions K of $\mathcal{O}_{X,x}$, and $s \in \mathcal{O}_{U,u}$ is non-zero. Let f be the minimal polynomial of u in K' over K. As $f(s) = g(s)s + c$ for some polynomial g over K and $c \in K^\times$, we can write c as a quotient of elements of S. Therefore s is invertible in K', and hence also in $K_{\overline{x}}$.

Step 3. We show that $\varphi\colon \underline{\mathbb{Z}}_X \to \underline{\mathbb{Z}}_\eta$ is an isomorphism.

We check this on stalks using Theorem 1.7. Let $\overline{x}$ be a geometric point of X, and write $\eta_{\overline{x}} = \eta \times_X \operatorname{Spec} \mathcal{O}_{X,\overline{x}}$. Obviously, we have $(\underline{\mathbb{Z}}_X)_{\overline{x}} = \mathbb{Z}$. Moreover, $(i_* \underline{\mathbb{Z}}_\eta)_{\overline{x}} = \Gamma(\eta_{\overline{x}}, \underline{\mathbb{Z}}_{\eta_{\overline{x}}})$. As $\eta_{\overline{x}}$ is the spectrum of $\mathcal{O}_{X,\eta} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,\overline{x}}$, which is a field by Step 2, it is then clear that $(i_* \underline{\mathbb{Z}}_\eta)_{\overline{x}} = \mathbb{Z}$. As we know the map $\varphi_{\overline{x}}$ is a map of rings, it must be the identity.

Step 4. Finally, we show that $R^1 i_* \underline{\mathbb{Z}}_\eta = 0$.

It suffices to show this at every stalk. As in Step 3, Theorem 1.7 states that for $x \in X$, and any geometric point $\overline{x}$ over x, we have

$$(R^1 i_* \underline{\mathbb{Z}}_\eta)_{\overline{x}} = \mathrm{H}^1(\eta_{\overline{x},\text{ét}}, \underline{\mathbb{Z}}_{\eta_{\overline{x}}}),$$

which is zero by Step 1.

Step 5. Hence $\mathrm{H}^1(X_{\text{ét}}, \underline{\mathbb{Z}}_X) = \mathrm{H}^1(X_{\text{ét}}, i_*\underline{\mathbb{Z}}_\eta) = \mathrm{H}^1(\eta_{\text{ét}}, \underline{\mathbb{Z}}_\eta) = 0$, where the second equality comes from the Leray spectral sequence. A direct proof of this isomorphism can be found in the appendix.

2 Appendix

The following is rather independent of the topology, but we state it for the étale topology anyway.

Lemma 2.1. *Let $f\colon Y \to X$ be a morphism of schemes, and let $\mathcal{F}$ be a sheaf in $\mathrm{Ab}(Y_{\acute{e}t})$ (or any other topology on Sch_Y). If $R^1 f_*\mathcal{F} = 0$, then $\mathrm{H}^1(X_{\acute{e}t}, f_*\mathcal{F}) = \mathrm{H}^1(Y_{\acute{e}t}, \mathcal{F})$.*

Proof. As stated in Section 1.2, this follows directly from the Leray spectral sequence, but here's a (hopefully fun) more direct proof of this fact, which only uses the long exact sequence of cohomology, and the fact (due to Grothendieck) that $R\Gamma(X_{\text{ét}}, Rf_*-) = R\Gamma(Y_{\text{ét}}, -)$, for a morphism $f\colon Y \to X$ of schemes.

Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$, and let $\mathcal{J}^\bullet$ be an injective resolution of $f_*\mathcal{I}^\bullet$, and denote its differentials by $d^i\colon \mathcal{J}^i \to \mathcal{J}^{i+1}$. As $R^1 f_*\mathcal{F} = 0$, it follows that $\mathcal{J}^\bullet$ is exact at $\mathcal{J}^1$. Hence we have a short exact sequence

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \mathcal{J}^0 \longrightarrow \ker d_1 \longrightarrow 0,$$

inducing a long exact sequence

$$0 \longrightarrow f_*\mathcal{F}(X) \longrightarrow \mathcal{J}^0(X) \longrightarrow \ker d_1(X) \longrightarrow \mathrm{H}^1(X, f_*\mathcal{F}) \longrightarrow \mathrm{H}^1(X, \mathcal{J}^0) = 0.$$

Therefore

$$\begin{aligned}\mathrm{H}^1(X_{\text{ét}}, f_*\mathcal{F}) &= \ker d_1(X) / \operatorname{im}\big(d_0(X)\big) = \mathrm{H}^1\big(\mathcal{J}^\bullet(X)\big) \\ &= \mathrm{H}^1\big(R\Gamma(X_{\text{ét}}, Rf_*\mathcal{F})\big) \\ &= \mathrm{H}^1\big(R\Gamma(Y_{\text{ét}}, \mathcal{F})\big) = \mathrm{H}^1(Y_{\text{ét}}, \mathcal{F}).\end{aligned}$$

□