The pro-étale site, part II

J. Jin

Last compile: 25th November, 2014; 18:10

Here's the section from last week that I didn't get to at the time.

1 Pro-étale coverings

Instead of what the terminology suggests, we will not use pro-étale morphisms of schemes to define the pro-étale site. Instead, we use the notion of a *weakly étale* morphism of schemes, which we define below.

1.1 Weakly étale morphisms

Definition 1.1. Let $f\colon Y \to X$ be a morphism of schemes. Then f is *weakly étale* if both f and the diagonal $\Delta_f\colon Y \to Y \times_X Y$ are flat.

We can characterise this in terms of the stalks as follows.

Proposition 1.2. *Let $f\colon Y \to X$ be a morphism of schemes. Then f is weakly étale if and only if for all $y \in Y$, the map* $\operatorname{Spec} \mathcal{O}_{Y,y} \to \operatorname{Spec} \mathcal{O}_{X,f(y)}$ *is weakly étale.*

Only requiring that the diagonal is flat already has the following nice consequence.

Lemma 1.3. *Let $f\colon Y \to X$ be a morphism of schemes such that $\Delta_f\colon Y \to Y \times_X Y$ is flat. Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. If $\mathcal{F}$ is flat over X, then $\mathcal{F}$ is flat over Y.*

Proof. Let $y \in Y$, let $A = \mathcal{O}_{X,f(y)}$ and $B = \mathcal{O}_{Y,y}$. Note that the functor

$$- \otimes_A \mathcal{F}_y \colon B\text{-}\mathrm{Mod} \to (B \otimes_A B)\text{-}\mathrm{Mod}$$

is exact. Moreover, $B \otimes_A B \to B$ is flat, so

$$- \otimes_{B \otimes_A B} B \colon (B \otimes_A B)\text{-}\mathrm{Mod} \to B\text{-}\mathrm{Mod}$$

is exact as well, so the composition of these two functors is exact as well.

This composition sends any B-module N to

$$(N \otimes_A \mathcal{F}_y) \otimes_{B \otimes_A B} B = N \otimes_B \mathcal{F}_y,$$

therefore $\mathcal{F}_y$ is flat over B, as desired. $\square$

Lemma 1.4. *Let $f\colon Y \to X$ and $g\colon Z \to Y$ be morphisms of schemes. Suppose that g is faithfully flat, and $\Delta_{fg}\colon Z \to Z \times_X Z$ is flat. Then $\Delta_f\colon Y \to Y \times_X Y$ is flat.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\Delta_{fg}} & Z \times_X Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g \times g} \\ Y & \xrightarrow[\Delta_f]{} & Y \times_X Y \end{array}$$

As $(g \times g)\Delta_{fg}$ is flat, and g is faithfully flat, it follows that Δ_f is flat. □

We have some basic properties.

Proposition 1.5.

(a) The base change of a weakly étale morphism is weakly étale.
(b) The composition of weakly étale morphisms is weakly étale.
(c) Any morphism of weakly étale S-schemes is weakly étale.
(d) A cofiltered limit of affine schemes that are weakly étale over an affine scheme, is weakly étale.
(e) The property of being weakly étale is fpqc local on the base.

Proof. The proof is left to the reader. Note that the proof of (c) uses Lemma 1.3. □

Property (e) above is the reason why we consider weakly étale morphisms instead of pro-étale ones, since (e) fails for pro-étale morphisms.

1.2 qc-coverings

There is one condition in the definition of an fpqc-covering that we would like to study on its own, as it will later appear as well in the definition of a pro-étale covering. Note: the terminology here is non-standard.

Definition 1.6. A morphism $\varphi\colon S' \to S$ of schemes is a *qc-covering* if for any quasi-compact open $U \subseteq S$ there exists a quasi-compact open $U' \subseteq S'$ such that $\varphi(U') = U$. A family $\{\varphi_i\colon S_i \to S\}$ of morphism of schemes is a *qc-covering* if $\varphi\colon \coprod_i S_i \to S$ is a qc-covering.

Note that qc-coverings always are (jointly) surjective.

Lemma 1.7.

(a) Let $\varphi_i\colon S_i' \to S_i$ be qc-coverings. Then $\varphi = \coprod_i \varphi_i\colon S' \to S$ is also a qc-covering, where $S = \coprod_i S_i$ and $S' = \coprod_i S_i'$.
(b) Let $\varphi\colon S' \to S$ and $\psi\colon S'' \to S'$ be qc-coverings. Then $\psi\varphi$ is a qc-covering.
(c) Let $\varphi\colon S' \to S$ be a qc-covering, and let $f\colon T \to S$ be a morphism of schemes. Then the base change $\varphi_T\colon T' \to T$, where $T' = T \times_S S'$, is a qc-covering.

The reason we haven't encountered this notion in the definition of e.g. the étale site is the following.

Proposition 1.8. *Any open surjective morphism $\varphi\colon S' \to S$ is a qc-covering.*

Proof. Let $U \subseteq S$ be a quasi-compact open. Cover its preimage $\varphi^{-1}U$ in S' by quasi-compact open subsets U_i'. As φ is surjective, the images $\varphi U_i'$ cover U. Since U is quasi-compact, there is a finite set J of indices i such that the $\varphi U_i'$ for $i \in J$ cover U. Set $U' = \bigcup_{i \in J} U_i'$. Then U' is a finite union of quasi-compact open subsets, hence quasi-compact, and $\varphi U' = U$. Hence φ is a qc-covering of S. □

Note that all faithfully flat morphisms that are locally of finite presentation are open and surjective, hence are qc-coverings.

1.3 The pro-étale sites

We can now define the coverings in the pro-étale site.

Definition 1.9. A morphism is a *pro-étale covering* if it is a weakly étale qc-covering. A family of morphisms $\{S_i \to S\}$ is a *pro-étale covering* if it is a qc-covering consisting of weakly étale morphisms.

Note that by Proposition 1.5 and Lemma 1.7, we can define the following (some set-theoretic issues aside).

Definition 1.10. Let S be a scheme.

- The *big pro-étale site* on S is the site $\mathrm{Sch}_{S,\text{pro-ét}}$ where the coverings are the pro-étale coverings.
- The *(small) pro-étale site* on S is the site $S_{\text{pro-ét}}$ which is as a category the full subcategory of Sch_S of weakly étale S-schemes, and in which the coverings are the qc-coverings.

As any pro-étale covering is an fpqc covering, it follows that the big and small pro-étale sites are subcanonical.

1.4 Examples of pro-étale coverings

First note that since strict henselisations of local rings are ind-étale by definition, we have the following family of examples.

Example 1.11. Let X be a scheme, and let $Y \in X_{\text{pro-ét}}$. Let $\overline{y}_1, \dots, \overline{y}_n$ be geometric points lying over distinct closed points $y_1, \dots, y_n$. Then

$$\Big(\coprod_i \operatorname{Spec} \mathcal{O}_{Y,\overline{y}_i}\Big) \sqcup \big(Y - \{y_1, \dots, y_n\}\big) \to Y$$

is a pro-étale covering of Y.

If we replace infinitely many points in the same way, the result is not a pro-étale covering as it is not a qc-covering.

Here's a more "uniform" family of examples.

Example 1.12. Let X be a scheme, and let $Y \in X_{\text{pro-ét}}$ be qcqs. Let G be the étale fundamental group of Y, and let S be a pro-finite continuous G-set.

Then S is a cofiltered limit $\lim_i S_i$ of finite continuous G-sets. For all i, let Y_i be the finite étale Y-scheme corresponding to the finite G-set S_i. This gives a cofiltered system of finite étale schemes over a qcqs scheme. Therefore its limit $Y \otimes S = \lim_i Y_i$ exists, and is a weakly étale Y-morphism. It is a pro-étale covering if and only if S is non-empty.

Using this construction, one can describe the affine objects in $(\operatorname{Spec} k)_{\text{pro-ét}}$ explicitly.

Proposition 1.13. *Let k be a field, and let $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Then the full subcategory of $(\operatorname{Spec} k)_{\text{pro-ét}}$ of affine objects is equivalent to the category of pro-finite continuous G-sets.*

Proof. We simply note that there is an obvious functor $X \mapsto X(k^{\mathrm{sep}})$ which is quasi-inverse to the functor $S \mapsto (\operatorname{Spec} k) \otimes S$. $\square$

Note that under this identification, the pro-étale coverings in $(\operatorname{Spec} k)_{\text{pro-ét}}$ consisting of affine objects correspond to $\{S_i \to S\}$ for which there exists a finite index set J such that $\coprod_j S_j \to S$ is surjective.

We will see in the next section that the collection of affine weakly étale k-schemes is enough to check the sheaf property on the pro-étale site with.

2 Generators for the pro-étale topology

In this section, we justify the fact that despite not (directly) using "pro-étale morphisms", this site is still called the pro-étale site. We will restrict our discussion mostly to the small pro-étale site. We follow [Bhatt-Scholze], rather than the Stacks Project.

Definition 2.1. An object U of $X_{\text{pro-ét}}$ is *pro-étale affine* if it is a cofiltered limit of affine schemes that are étale over X.

Note that a pro-étale affine object of $X_{\text{pro-ét}}$ is indeed affine, and pro-étale over X. Denote the full subcategory of $X_{\text{pro-ét}}$ of pro-étale affines by $X^{\text{aff}}_{\text{pro-ét}}$. It becomes a site when taking pro-étale surjections as coverings. We will show that $X_{\text{pro-ét}}$ is *generated* by $X^{\text{aff}}_{\text{pro-ét}}$; the precise statement is the following.

Proposition 2.2. *Every $Y \in X_{\text{pro-ét}}$ admits a covering $\{U_i \to Y\}$ with $U_i \in X^{\text{aff}}_{\text{pro-ét}}$.*

In order to prove this, we study the notion of ind-étale ring maps more closely, and study its relation with weakly étale ring maps.

2.1 Ind-étale and weakly étale algebras

Definition 2.3. Let A be a ring. An A-algebra B is *ind-étale* if B is isomorphic to a filtered colimit of étale A-algebras.

The basic properties hold for ind-étale ring maps.

Proposition 2.4. *(a) Let $A \to B$ and $A \to A'$ be morphisms of rings, and write $B' = B \otimes_A A'$. If $A \to B$ is ind-étale, then so is $A' \to B'$.*
(b) Let $A \to B$ and $B \to C$ be morphisms of rings. If $A \to B$ and $B \to C$ are ind-étale, then so is their composition $A \to C$.
(c) Let $A \to B$ and $B \to C$ be morphisms of rings. If $A \to B$ and the composition $A \to C$ are ind-étale, then so is $B \to C$.
(d) Any filtered colimit of ind-étale A-algebras is an ind-étale A-algebra.

Note that by Proposition 1.5.(d), ind-étale ring maps are weakly étale. Now the following theorem implies Proposition 2.2.

Theorem 2.5 ([Bhatt-Scholze, Thm. 2.3.4])**.** *Let $A \to B$ be a weakly étale morphism of rings. Then there exists a faithfully flat ind-étale morphism $B \to C$ of rings such that $A \to C$ is ind-étale.*

2.2 Characterising sheaves on the pro-étale site

The fact that $X_{\text{pro-ét}}$ is generated by $X^{\text{aff}}_{\text{pro-ét}}$ has the following consequence.

Let $X_{\text{pro-ét,aff}}$ denote the subcategory of all affine objects of $X_{\text{pro-ét}}$. Note that this category is not the same as $X^{\text{aff}}_{\text{pro-ét}}$. We give it the structure of a site by taking all pro-étale surjections as coverings.

Proposition 2.6. *Let $\mathcal{F}$ be a presheaf (of sets) on $X_{\text{pro-ét}}$. Then $\mathcal{F}$ is a sheaf on $X_{\text{pro-ét}}$ if and only if*
(a) $\mathcal{F}$ is a Zariski sheaf,
(b) for all coverings $V \to U$ with $V \in X_{\text{pro-ét,aff}}$, the diagram

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is an equaliser diagram.

Proof. This proof is inspired by [SP,022H]. If $\mathcal{F}$ is a sheaf, then there is nothing to prove.

So assume (a) and (b). By (a), for all $U = \coprod_i U_i$, we have $\mathcal{F}(U) = \prod_i \mathcal{F}(U_i)$. We shall use this fact tacitly.

We first show that $\mathcal{F}$ is separated. Let $V \to U$ be a pro-étale covering with V arbitrary and U affine. Note that there is a cover $W \to U$ with $U, W \in X_{\text{pro-ét,aff}}$ refining $V \to U$, by Proposition 2.2 and as U is affine. Therefore $\mathcal{F}(U) \to \mathcal{F}(W)$ is injective by (b), so so is $\mathcal{F}(U) \to \mathcal{F}(V)$.

Now let $V \to U$ be an arbitrary pro-étale covering. Let $\{U_i\}$ be an affine open covering of U, and write $V_i = V \times_U U_i$. Then the composition $\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_i \mathcal{F}(V_i)$ is injective, therefore $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective as well. Hence $\mathcal{F}$ is separated.

Now we show that $\mathcal{F}$ is a sheaf. First let $f\colon V \to U$ be an arbitrary pro-étale covering with U affine. Again, there is a cover $W \to U$ with $W \in X_{\text{pro-ét,aff}}$ refining $V \to U$. Behold the following diagram.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, equal] & \mathcal{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[ddl, equal] & \mathcal{F}(V \times_U V) \arrow[d] \\
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(W) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{F}(W \times_U W) \\
\mathcal{F}(V) \arrow[r] & \mathcal{F}(V \times_U W) &
\end{tikzcd}$$

Note that this diagram is commutative (for a suitable choice of one of each pair of parallel arrows) as it arises from a commutative diagram of schemes.

By assumption the middle row is an equaliser diagram. We show that the top row is an equaliser diagram as well.

Suppose that $s \in \mathcal{F}(V)$ is such that its two images to the right are equal. Then the same holds for $s|_W$, so by (b) there exists a unique $t \in \mathcal{F}(U)$ such that $t|_W = s|_W$. Note that $\mathcal{F}(V) \to \mathcal{F}(V \times_U W)$ is injective. Both $t|_V$ and s are mapped to the same element by this map, therefore $t|_V = s$, and t is the unique element of $\mathcal{F}(U)$ with this property.

Now suppose that $V \to U$ is an arbitrary pro-étale covering. Let $\{U_i\}$ be an affine open covering of U, and let $U_{ij} = U_i \cap U_j$. Moreover, write $V_i = V \times_U U_i$ and $V_{ij} = V \times_U U_{ij}$. Behold the following diagram.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{F}(V \times_U V) \arrow[d] \\
\prod_i \mathcal{F}(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \mathcal{F}(V_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \mathcal{F}(V_i \times_{U_i} V_i) \\
\prod_{i,j} \mathcal{F}(U_{ij}) \arrow[r] & \prod_{i,j} \mathcal{F}(V_{ij}) &
\end{tikzcd}$$

Again, this diagram is commutative, as it comes from a commutative diagram of schemes.

By assumption, and by the previous step, all pairs of parallel arrows except for the one in the top row belong to an equaliser diagram. We show that the top row is an equaliser diagram as well.

Suppose that $s \in \mathcal{F}(V)$ is such that its two images to the right are equal. Then the same holds for $s|_{V_i} \in \mathcal{F}(V_i)$ for all i. Therefore there exists unique $t_i \in \mathcal{F}(U_i)$ such that $t_i|_{V_i} = s|_{V_i}$ for all i. Note that the map $\prod_{i,j} \mathcal{F}(U_{ij}) \to \prod_{i,j} \mathcal{F}(V_{ij})$ is injective. Therefore the two downward images of each t_i are equal. Hence there exists a unique $t \in \mathcal{F}(U)$ such that $t|_{U_i} = t_i$. Now $t|_V$ and s map to the same element of $\prod_i \mathcal{F}(V_i)$, so $t|_V = s$. We conclude that $\mathcal{F}$ is a sheaf on $X_{\text{pro-ét}}$. $\square$

Note that this is slightly weaker than [Bhatt-Scholze, Lem. 4.2.6].

Corollary 2.7. *Let X be a scheme. Then the restriction functor $\mathrm{Sh}(X_{\text{pro-ét}}) \to \mathrm{Sh}(X^{\text{aff}}_{\text{pro-ét}})$ is an equivalence of categories.*

Proof. It remains to show that the restriction functor $\mathrm{Sh}(X_{\text{pro-ét,aff}}) \to \mathrm{Sh}(X^{\text{aff}}_{\text{pro-ét}})$ is an equivalence of categories. We explicitly give a quasi-inverse, but leave the proof that it is indeed a quasi-inverse to the reader.

Let $\mathcal{F} \in \mathrm{Sh}(X^{\text{aff}}_{\text{pro-ét}})$. Define $\overline{\mathcal{F}}$ as follows. For $U \in X_{\text{pro-ét,aff}}$, set $\overline{\mathcal{F}}(U) = \lim_V \mathcal{F}(V)$, where V runs through the $V \to U$ in $X_{\text{pro-ét,aff}}$ with $V \in X^{\text{aff}}_{\text{pro-ét}}$.

Note that $\overline{\mathcal{F}}$ is a presheaf as for every $U' \to U$ in $X_{\text{pro-ét,aff}}$, the limit taken in $\overline{\mathcal{F}}(U')$ is a subset of that taken in $\overline{\mathcal{F}}(U)$. We now show that $\overline{\mathcal{F}}$ is a sheaf.

Let $U' \to U$ be a covering in $X_{\text{pro-ét,aff}}$. Then for any $V \to U$ with $V \in X^{\text{aff}}_{\text{pro-ét}}$, we get a covering $V' = V \times_U U' \to V$ in $X^{\text{aff}}_{\text{pro-ét}}$. Therefore $\overline{\mathcal{F}}(U) \to \overline{\mathcal{F}}(U')$ is injective, i.e. $\overline{\mathcal{F}}$ is separated.

If $s \in \overline{\mathcal{F}}(U')$ is such that its two images in $\overline{\mathcal{F}}(U' \times_U U')$ coincide, then the same holds for $s_{V'} \in \mathcal{F}(V')$ for all $V \to U$ with $V \in X^{\text{aff}}_{\text{pro-ét}}$. Hence for all V, there exists a unique $t_V \in \mathcal{F}(V)$ with $t_V|_{V'} = s_{V'}$. The t_V are compatible with the transition maps, hence define a unique element $t \in \overline{\mathcal{F}}(U)$. For any $V' \to U'$ with $V' \in X^{\text{aff}}_{\text{pro-ét}}$ (so not just the ones obtained by base change), we have $s_{V'}|_{V''} = t_{V'}|_{V''}$ (both equal to $s_{V''}$), therefore $s_{V'} = t_{V'}$, so t maps to $s \in \overline{\mathcal{F}}(U')$. Therefore $\overline{\mathcal{F}}$ is a sheaf on $X_{\text{pro-ét,aff}}$. □

2.3 Examples of sheaves

Proposition 2.8 ([Bhatt-Scholze, Lem. 4.2.12]). *Let T be any topological space, and let X be a scheme. Consider the presheaf $\mathcal{F}_T$ on $X_{\text{pro-ét}}$ mapping U to $\mathrm{Hom}_{\mathrm{Top}}(U, T)$. Then $\mathcal{F}_T$ is a pro-étale sheaf.*

Proof. It is clear that $\mathcal{F}_T$ is a Zariski sheaf. Therefore it remains to check that the sheaf property holds for $\mathcal{F}_T$ for every pro-étale covering $\varphi\colon V \to U$ in $X_{\text{pro-ét,aff}}$.

In other words, we need to show that if $g\colon V \to T$ is a continuous map such that $g\pi_1 = g\pi_2\colon V \times_U V \to T$, then there exists a unique continuous map $f\colon U \to T$ such that $g = f\varphi$. As φ is surjective, there is at most one such f. By checking on fibres, we see that the natural map $|V \times_U V| \to |V| \times_{|U|} |V|$ is surjective, therefore there is a unique map $f\colon U \to T$ such that $g = f\varphi$. Note that φ is universally submersive by [SP,02JY], therefore a subset $U' \subseteq U$ is open if and only if $\varphi^{-1}U$ is. Therefore f is continuous, as desired. □

3 Exactness of sequences of sheaves

We now give criteria for a sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ of abelian pro-étale sheaves to be exact.

3.1 Classical stalks

We repeat the construction of stalks, in the same way as for the étale site, but then for the pro-étale site.

Definition 3.1. Let $\overline{x} \in X$ be a geometric point. A *pro-étale neighbourhood* $(U, \overline{u})$ of $\overline{X}$ is a weakly étale morphism $(U, \overline{u}) \to (X, \overline{x})$ of geometrically pointed schemes.

The category of pro-étale neighbourhoods turns out to be cofiltered again, and we can define the *stalk* $\mathcal{F}_{\overline{x}}$ of a pro-étale sheaf $\mathcal{F}$ on X in the usual way, as the colimit of all $\mathcal{F}(U)$ for all pro-étale neighbourhoods $(U, \overline{u})$ of $\overline{x}$. As usual, the stalk only depends on the image of the geometric point.

We can define this stalk in an alternative way.

Proposition 3.2 ([SP,0993]). *Let X be a scheme, let $\overline{x}$ be a geometric point of X, and let $\mathcal{F}$ be a pro-étale sheaf on X. Then* $\operatorname{Spec}(\mathcal{O}_{X,\overline{x}}) \in X_{\text{pro-ét}}$, *and there is a canonical isomorphism*

$$\mathcal{F}\big(\operatorname{Spec}(\mathcal{O}_{X,\overline{x}})\big) = \mathcal{F}_{\overline{x}},$$

which is functorial in $\mathcal{F}$.

This follows from a theorem of Olivier which states that any local morphism of local rings from a strictly henselian ring is an isomorphism.

Classical stalks are not enough for our purposes, though, as the following example shows.

Example 3.3. Let A be an abelian group, and let X be the spectrum of a separably closed field. By Proposition 2.8, both $\mathcal{F}$ given by $\mathcal{F}(U) = \operatorname{Hom}_{\mathrm{Top}}(U, A)$ and $\mathcal{G}$ given by $\mathcal{G}(U) = \operatorname{Hom}_{\mathrm{Set}}(U, A)$ are pro-étale sheaves on X. Consider their quotient $\mathcal{Q} = \mathcal{G}/\mathcal{F}$. Then its unique stalk $\mathcal{Q}(X)$ is zero.

However, $\mathcal{Q}$ is non-zero, since otherwise $\mathcal{F} = \mathcal{G}$, and we have $\mathcal{F}(U) \neq \mathcal{G}(U)$ for any $U \in X^{\mathrm{aff}}_{\text{pro-ét}}$ non-discrete.

3.2 w-contractible rings

A way to remedy the failure in the previous example is the use of so-called *w-contractible* rings. As we will see below, w-contractibility is like an injectivity condition.

Definition 3.4. A ring A is *w-contractible* if every faithfully flat, weakly étale ring map $A \to B$ has a retraction (i.e. an A-algebra map $B \to A$).

This class of objects extends that of étale local rings.

Lemma 3.5. *Let A be a strictly henselian local ring. Then A is w-contractible.*

Proof. Let $A \to B$ be weakly étale and faithfully flat. Then there exists a prime $\mathfrak{q}$ of B such that its inverse image is maximal in A. By Olivier's theorem, the composition $A \to B \to B_{\mathfrak{q}}$ is an isomorphism of A-algebras. □

Proposition 3.6. *Let A be a ring, and let $T = \operatorname{Spec} A$. The following are equivalent.*

(a) A is w-contractible;

(b) for any pro-étale covering of T, there exists a finite partition of open and closed subschemes of T refining it;

(c) the following holds;

- *T is* w-local *(i.e. the set $T_0 \subseteq T$ of closed points is closed, and the map $T_0 \to \pi_0(T)$ to the set of connected components of T is a bijection);*
- *T_0 is* extremally disconnected *(i.e. the closure of an open subset of T_0 is open);*
- *for all maximal ideals $\mathfrak{m} \subseteq A$, its localisation $A_{\mathfrak{m}}$ is strictly henselian.*

Proof. We show that (a) and (b) are equivalent.

Suppose A is w-contractible, and let $\{U_i \to T\}$ be a pro-étale covering. By definition, we can refine this by a finite pro-étale covering $\{U'_j \to T\}$, we hence get a weakly étale surjective map $f\colon \coprod_j U'_j \to T$ of affine schemes. Therefore there exists a section $s\colon T \to \coprod_j U'_j$ of f. Setting $T_j = s^{-1}U'_j$ gives us the desired partition.

Suppose that for all pro-étale coverings of T, there exists a finite partition of open and closed subschemes of T refining it. Let $U \to T$ be a weakly étale surjective map of affine schemes. Let $\{T_i \to T\}$ be a finite partition of open and closed subschemes refining $U \to T$, i.e. there exists a morphism $T = \coprod_i T_i \to U$, and this map is a section of $U \to T$. Hence A is w-contractible.

For (c), see [SP,0982]. □

As usual for objects satisfying an injectivity condition, the existence of enough of them is hard, if true at all. Therefore we don't prove the following theorem.

Theorem 3.7 ([Bhatt-Scholze, Lem. 2.4.9])**.** *Let A be a ring. Then there exists a faithfully flat, ind-étale A-algebra B which is w-contractible.*

This gives us a criterion to determine whether sequences of sheaves are exact or not.

Lemma 3.8. *Let X be a scheme, and let $U \in X_{\text{pro-ét,aff}}$ be a w-contractible object. Then U is* weakly contractible*, i.e. for any surjective map $\mathcal{F} \to \mathcal{G}$ of sheaves on $X_{\text{pro-ét}}$, the section map $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective.*

Proof. Let $s \in \mathcal{G}(U)$ be a section. Then there exists a pro-étale covering $\{V_i \to U\}$ of U such that each $s|_{V_i}$ has a pre-image in $\mathcal{F}(V_i)$. By w-contractibility, this covering admits a refinement of the form $U = \coprod_i U_i$, so therefore every $s|_{U_i}$ has a pre-image t_i in $\mathcal{F}(U_i)$. But now $\mathcal{F}(U) = \prod_i \mathcal{F}(U_i)$ and $\mathcal{G}(U) = \prod_i \mathcal{G}(U_i)$, so (t_i) is a section of $\mathcal{F}(U)$ mapping to $s \in \mathcal{G}(U)$. ☐

Theorem 3.9. *Let X be a scheme, and let $\mathcal{O}$ be a sheaf of rings on $X_{\text{pro-ét}}$. A sequence*

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \tag{1}$$

of $\mathcal{O}$-modules is exact if and only if for all w-contractible objects U of $X_{\text{pro-ét,aff}}$, the sequence

$$\mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \tag{2}$$

is exact.

Remark 3.10. The argument that follows is essentially a formal argument for any site that "has enough quasi-compact weakly contractible objects".

Proof. Of course, if (1) is exact, by Lemma 3.8, then for all $U \in X^{\text{aff}}_{\text{pro-ét}}$ w-contractible the sequence (2) is exact. So suppose that for all $U \in X_{\text{pro-ét,aff}}$ w-contractible, (2) is exact.

As (2) is a complex for all $U \in X_{\text{pro-ét,aff}}$ w-contractible, then as every object of $X_{\text{pro-ét}}$ admits a covering by pro-étale affines by Proposition 2.2, hence also by w-contractible objects by Theorem 3.7, it follows that (1) is a complex as well. So let $V \in X_{\text{pro-ét}}$, let $s \in \ker\big(\mathcal{G}(V) \to \mathcal{H}(V)\big)$. We show that it is in the image of $\mathcal{F} \to \mathcal{G}$.

Let $\{U_i \to V\}$ be a pro-étale covering consisting of w-contractible objects. By assumption, $s|_{U_i}$ has a pre-image in $\mathcal{F}(U_i)$. Therefore it lies in the image of $\mathcal{F} \to \mathcal{G}$. Hence (1) is exact. ☐

We finish by stating some corollaries of this.

Corollary 3.11. *Let X be a scheme, and let $\mathcal{O}$ be a sheaf of rings on $X_{\text{pro-ét}}$. Let $\mathcal{F}$ be any $\mathcal{O}$-module. Then $\mathrm{H}^p(U_{\text{pro-ét}}, \mathcal{F}) = 0$ for all w-contractible $U \in X_{\text{pro-ét,aff}}$ and all $p \geq 1$.*

Corollary 3.12. *Let X be a scheme, and let $\mathcal{O}$ be a sheaf of rings on $X_{\text{pro-ét}}$. Let*

$$\cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0$$

be a cofiltered diagram of $\mathcal{O}$-modules in which the transition maps are surjective. Then $\lim \mathcal{F}_i \to \mathcal{F}_0$ is surjective as well. (In other words, the category of $\mathcal{O}$-modules is replete*.)*

Corollary 3.13. *Let X be a scheme, and let $\mathcal{O}$ be a sheaf of rings on $X_{\text{pro-ét}}$. Then products are exact on the category of $\mathcal{O}$-modules. (I.e. the product of exact sequences of $\mathcal{O}$-modules is exact.)*

4 References

[Bhatt-Scholze] B. Bhatt, P. Scholze, *The pro-étale topology for schemes*, arXiv:1309.1198